More on linear transformations

Last class we discussed the following list of functions. (Some are linear, but others are not.)

Examples: functions $f : \mathbb{R} \to \mathbb{R}$

1. $f_1(x) = 2x$
2. $f_2(x) = 2x + 1$
3. $f_3(x) = x^2$
4. $f_4(x) = \cos x$

Examples: functions $g : \mathbb{R}^2 \to \mathbb{R}^2$

1. $g_1(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$
2. $g_2(x_1, x_2) = (\cos(x_1 + x_2), x_1 + x_2^2)$

Examples: functions h defined on $\mathbb{R}^3$

1. $h_1(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_2 + x_3)$
2. $h_2(x_1, x_2, x_3) = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

One way to show that a transformation is linear is to verify the two conditions of linearity directly, but there is an easier way to see that these transformations are linear.

Important class of examples: Given an $m \times n$ matrix $\mathbf{A}$, then we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by the equation

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

We know that T is a linear transformation because the matrix-vector product satisfies the necessary conditions.

Example. Let

$$\mathbf{G} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Then

$$\mathbf{G}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

There is a link on the course web page to a java program called `Matrix Machine` that lets you investigate the mapping properties of various 2×2 matrix transformations. Try the following three matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Theorem. Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $m \times n$ matrix. This matrix $\mathbf{A}$ is called the *standard matrix representation* of T.

Why? Let's make two observations using the "standard basis" of $\mathbb{R}^n$.

Definition. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \ldots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

1. If we know the images $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ for all of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then we can calculate $T(\mathbf{x})$ for any $\mathbf{x}$ in $\mathbb{R}^n$.

2. If

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{array} \right],$$

then

$$\mathbf{A}\mathbf{x} = x_1 T(\mathbf{e}_1) + \ldots + x_n T(\mathbf{e}_n).$$

Example. What is the matrix representation of the linear transformation that rotates $\mathbb{R}^2$ by 45° ($\pi/4$ radians) around the origin?

Examples of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$

1. rotations

2. reflections

3. contractions and expansions

4. shears

5. projections