A little more on linear transformations

Last class we talked about rotations and reflections of $\mathbb{R}^2$ to $\mathbb{R}^2$. There are other types of linear transformations as well.

3. contractions and expansions

4. shears

5. projections

Definitions. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. We say that T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.
2. We say that T is one-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Examples. Let

$$\mathbf{P}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the only solution to $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ is the trivial solution.

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\mathbf{A}$ be its standard matrix representation. Then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $\mathbf{A}$ span $\mathbb{R}^m$, and
2. T is one-to-one if and only if the columns of $\mathbf{A}$ are linearly independent.