Coordinates relative to a basis

Last class we saw that a basis for a vector space produces a coordinate system for that space.

Theorem. (Unique Representation Theorem) Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then every vector $\mathbf{v}$ in V can be represented uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

The scalars c_1, ..., c_n are called the coordinates of $\mathbf{v}$ relative to the basis B.

Example. One basis of $\mathbb{P}_3$ is $\{x^3 + 1, x, x^2, 4\}$. The coordinates of $2x^3 - x^2$ relative to this basis are $(2, 0, -1, -\frac{1}{2})$ because

$$2x^3 - x^2 = 2(x^3 + 1) - x^2 - (\tfrac{1}{2})4.$$

Why are coordinates relative to a given basis unique?

The same vector has different coordinates relative to different bases.

Example. Consider the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$. What are its coordinates relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and what are its coordinates relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}?$$

Notation. Given the representation $\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ relative to the basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then the coordinates can be viewed as a vector in $\mathbb{R}^n$. This vector is denoted

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Example. What is the coordinate vector for $2x^3 - x^2$ relative to the basis $B = \{x^3 + 1, x, x^2, 4\}$ of $\mathbb{P}_3$?

Change of coordinates matrix

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$, then the B-coordinates of a vector $\mathbf{x}$ are related to the standard coordinates by the equation

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\mathbf{x} = \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \mathbf{P}_B \, [\mathbf{x}]_B$$

where $\mathbf{P}_B$ is the matrix

$$\mathbf{P}_B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \quad \ldots \quad & \mathbf{b}_n \end{array} \right].$$

Since $\mathbf{P}_B$ is invertible, we also have $[\mathbf{x}]_B = (\mathbf{P}_B)^{-1}\,\mathbf{x}$.

Example. We can double check our computation of the B-coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$ relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

using these equations.

For any vector space V with basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, the B-coordinates define a nice linear transformation from V onto $\mathbb{R}^n$. The map is defined by

$$\mathbf{v} \mapsto [\mathbf{v}]_B.$$

Theorem. The coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_B$ is a one-to-one linear transformation that maps V onto $\mathbb{R}^n$.

Definition. A one-to-one linear transformation that maps V onto W is called an isomorphism.

From the vector space point of view, two isomorphic vector spaces have the same structure.

Example. For what n is $\mathbb{R}^n$ isomorphic to $\mathbb{P}_3$?

Example. For what n is $\mathbb{R}^n$ isomorphic to $M_{2\times 3}$?

The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

Definition. A vector space V is finite dimensional if it contains a finite spanning set. Otherwise, V is said to be infinite dimensional.

Example. $\mathbb{R}^n$ is spanned by the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Therefore, it is finite dimensional.

Example. $\mathbb{P}$ is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)

Theorem. Let V be a vector space. Any finite spanning set for V has at least as many elements as any linearly independent subset of V.