The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

Definition. A vector space V is finite dimensional if it contains a finite spanning set. Otherwise, V is said to be infinite dimensional.

Example. $\mathbb{R}^n$ is spanned by the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Therefore, it is finite dimensional.

Example. The vector space $\mathbb{P}_3$ of all polynomial functions whose degree is at most three is spanned by the basis $\{1, x, x^2, x^3\}$. Therefore, it is finite dimensional.

Example. $\mathbb{P}$ is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)

Theorem. Let V be a vector space. Any finite spanning set for V has at least as many elements as any linearly independent subset of V.

Corollary. Any two bases of a finite-dimensional vector space V have the same number of elements.

Definition. The dimension of a finite-dimensional vector space V is the number of elements in any basis of V. This nonnegative integer is denoted $\dim V$.

Examples.

1. $\dim \mathbb{R}^n = n$

2. Let P be the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

   Hence, $\dim P = 2$.

3. $\dim \mathbb{P}_3 = 4$

4. $\dim M_{2\times 3} = 6$

Here are a couple of other consequences of the notion of dimension.

Theorem. If $\dim V = n$, then any set in V with more than n vectors must be linearly dependent.

Theorem. If H is a subspace of V, then $\dim H \leq \dim V$. In fact, any basis of H can be expanded to a basis of V.

Suppose that $\mathbf{A}$ is an $m \times n$ matrix. How can we determine the dimensions of Col $\mathbf{A}$ and Nul $\mathbf{A}$?

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}.$$

What relationship is there between the dimensions of Col $\mathbf{A}$ and Nul $\mathbf{A}$?