Determinants

We start with a recursive definition of the determinant.

Definition. The determinant of a 1×1 matrix $[a_{11}]$ is a_{11}.

Now we define the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices.

Definition. Given an $n \times n$ matrix $\mathbf{A}$, the ijth minor $\mathbf{A}_{ij}$ of $\mathbf{A}$ is the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by eliminating the ith row and jth column. The ijth cofactor of $\mathbf{A}$ is

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}.$$

Example. Compute the cofactors of the third column of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}.$$

Definition/Theorem. If $\mathbf{A}$ is an $n \times n$ matrix, the determinant of $\mathbf{A}$ can be computed using cofactor expansion along the ith row by

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

or by cofactor expansion along the jth column by

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Any row or any column yields the same result.

Example. Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}$$

by cofactor expansion along the third column.

Note that we get the familiar formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Is there a way to define $\det \mathbf{A}$ without recursion?

How do we go about computing $\det \mathbf{A}$?

One type of matrix is perfectly suited for cofactor expansion.

Theorem. If $\mathbf{A}$ is a triangular matrix, then $\det \mathbf{A}$ is the product of its entries along the main diagonal.

In order to gain some insight into how we will compute determinants in general, let's calculate the determinants of all elementary 3×3 matrices.