The diagonalization problem

A matrix $\mathbf{A}$ is diagonalizable if there exists a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

“Diagonalizing” a matrix has many applications. One is geometric. As we shall see, the matrix

$$\mathbf{B} = \begin{bmatrix} \frac{31}{45} & \frac{19}{45} \\ -\frac{19}{90} & \frac{119}{90} \end{bmatrix}$$

is diagonalizable, and the corresponding diagonal matrix is

$$\mathbf{A} = \begin{bmatrix} \frac{9}{10} & 0 \\ 0 & \frac{10}{9} \end{bmatrix}.$$

There are animations on the course web page that illustrate how the matrix $\mathbf{B}$ transforms the plane in a way that is “similar” to the diagonal matrix $\mathbf{A}$.

Diagonalizing a matrix is a special case of the similarity problem.

Definition. Two square matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Note: We have already done two exercises related to similarity—Section 2.2 #18 and Section 3.2 #34.

Theorem. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are similar matrices. Then $\mathbf{A}$ and $\mathbf{B}$

1. have the same characteristic polynomial and consequently the same eigenvalues, and
2. the same geometric multiplicities for each eigenvalue.

What does this theorem say about matrices that can be diagonalized? In other words, if a matrix **A** can be diagonalized, what must **A** and **D** have in common?

Example. What can you say about a matrix **A** that is similar to the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}?$$

For an arbitrary matrix **A**, what can be said about it if it is diagonalizable?

For example, can

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

be diagonalized?