The dot product

Definition. Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in $\mathbb{R}^n$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

Example. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \pi \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 5 \\ -3\sqrt{2} \\ 2 \end{bmatrix}.$$

Remarks.

1. The dot product $\mathbf{u} \cdot \mathbf{v}$ can be viewed as matrix multiplication, that is, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.
2. Note that $\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w}$ is undefined.
3. Note that $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ involves scalar multiplication.

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times p$ matrix. Write $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{array}\right] \qquad \text{and} \qquad \mathbf{B} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_p \end{array}\right].$$

Let's interpret the product $\mathbf{AB}$ in terms of the dot product.

Theorem 1. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u} \geq 0$, and

(e) $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Definition. The length (or norm) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}}.$$

Note that $||r\mathbf{v}|| = |r|\,||\mathbf{v}||$.

Given $\mathbf{v} \neq \mathbf{0}$, we normalize $\mathbf{v}$ by computing the vector $\mathbf{u} = \dfrac{1}{||\mathbf{v}||}\mathbf{v}$.

If we think of two vectors $\mathbf{u}$ and $\mathbf{v}$ as points in $\mathbb{R}^n$, then we define the distance between $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

What about angles? Let's start with right angles.

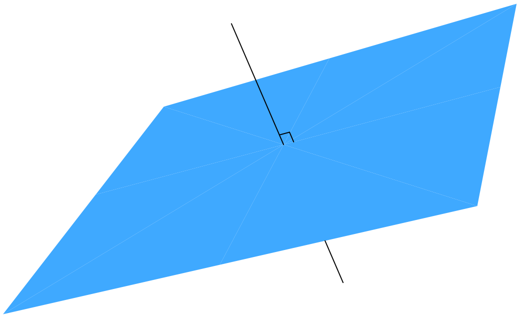
Definition. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$.

We can use the Law of Cosines to derive a more exact relationship between angles and the dot product.

Orthogonal complements

Given a subspace S of $\mathbb{R}^n$, we can consider the set of all vectors that are orthogonal to all vectors in S. For example, a plane through the origin in $\mathbb{R}^3$ can be described by one homogeneous linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$



Definition. Given a subspace S of $\mathbb{R}^n$, its orthogonal complement $S^\perp$ is the set

$$\{\mathbf{v} \mid \mathbf{v}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \in S\}.$$

Examples.

1. The orthogonal complement of a line through the origin in $\mathbb{R}^3$ is a plane through the origin.

2. The orthogonal complement of a plane through the origin in $\mathbb{R}^3$ is a line through the origin.

3. Consider an $m \times n$ matrix

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{array}\right].$$

What can you say about a vector $\mathbf{v}$ in $\mathbb{R}^n$ that is orthogonal to all of the rows of $\mathbf{A}$?

Theorem. Let S be a subspace of $\mathbb{R}^n$ and let $S^\perp$ be its orthogonal complement. Then

1. $S^\perp$ is a subspace of $\mathbb{R}^n$,
2. $\dim(S^\perp) = n - \dim(S)$,
3. $(S^\perp)^\perp = S$, and
4. every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written uniquely as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2,$$

where $\mathbf{v}_1$ is in S and $\mathbf{v}_2$ is in $S^\perp$.