More on orthogonal projection

Recall the Orthogonal Decomposition Theorem from last class.

Theorem. (Orthogonal Decomposition Theorem)

1. Each vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written uniquely as

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp},$$

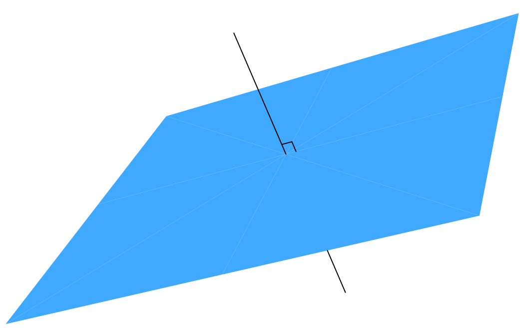
   where $\mathbf{w}$ is in W and $\mathbf{w}^{\perp}$ is in $W^{\perp}$.

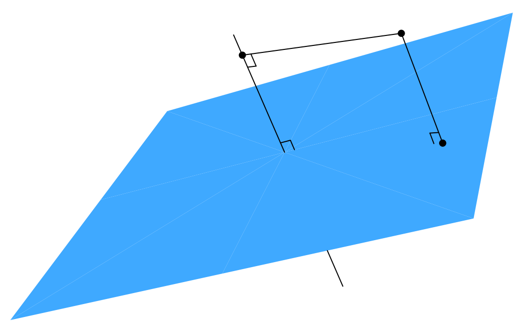
2. Given an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of W, then

$$\text{proj}_W \mathbf{v} \equiv \mathbf{w} = \left(\frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 + \ldots + \left(\frac{\mathbf{v} \cdot \mathbf{w}_k}{\mathbf{w}_k \cdot \mathbf{w}_k}\right) \mathbf{w}_k$$

   and $\mathbf{w}^{\perp} = \mathbf{v} - \mathbf{w}$.

Note: Since the two vectors $\mathbf{w}$ and $\mathbf{w}^{\perp}$ are unique, they do not depend on the orthogonal basis of W that we use to compute them.





Important consequence: If we want to find the distance of a vector $\mathbf{v}$ to a subspace W, then we compute

$$||\mathbf{w}^{\perp}|| = ||\mathbf{v} - \text{proj}_W \mathbf{v}||.$$

Example. Find the point closest to

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

in the subspace W spanned by the two vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Theorem. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for a subspace W, then

$$\text{proj}_W \mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + \ldots + (\mathbf{v} \cdot \mathbf{u}_k)\mathbf{u}_k.$$

If

$$\mathbf{U} = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \quad\ldots\quad & \mathbf{u}_k \end{array} \right],$$

then $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T \mathbf{v}$.

Example. Let's repeat the previous calculation using the projection matrix. Let

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{4}{\sqrt{26}} \\ -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{26}} \\ -\frac{1}{\sqrt{10}} & 0 \\ \frac{2}{\sqrt{10}} & \frac{3}{\sqrt{26}} \end{bmatrix}.$$

The Gram-Schmidt Process

This procedure produces an orthogonal (or orthonormal) basis from a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ of a subspace W. It is an inductive procedure.

We work with the subspaces

$$S_l = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_l\}.$$

The orthogonal basis for W based on this procedure applied to this basis is denoted $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$.
3. Let $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \dfrac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$.

etc.

Example. Apply the Gram-Schmidt process to the basis

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Example. Let's use these ideas to find the projection matrix **P** for orthogonal projection onto the plane $x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$.

What are the eigenvalues and eigenspaces of **P**? (No computation required)