

## More on the Gram-Schmidt Process

Recall that this procedure produces an orthogonal basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_l\}$  for a subspace  $W$  from an arbitrary basis  $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ .

1. Let  $\mathbf{v}_1 = \mathbf{x}_1$ .
2. Let  $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ .
3. Let  $\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$ .

etc.

**Example.** Apply the Gram-Schmidt process to the basis

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

## Projection matrices

We discussed projection matrices briefly last class. In particular, we discussed the following theorem.

**Theorem.** Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  be an orthonormal basis for a subspace  $W$  of  $\mathbb{R}^n$ . Form the  $n \times k$  matrix

$$\mathbf{U} = \left[ \begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k \end{array} \right].$$

Then  $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T \mathbf{v}$ .

The matrix  $\mathbf{P} = \mathbf{U}\mathbf{U}^T$  is called the *projection matrix* for the subspace  $W$ . It does not depend on the choice of orthonormal basis.

**Example.** Let's compute the projection matrix  $\mathbf{P}$  for orthogonal projection onto the plane  $x_1 + x_2 - x_3 = 0$  in  $\mathbb{R}^3$ .

What are the eigenvalues and eigenspaces of  $\mathbf{P}$ ? (No computation required)

What if we do not start with an orthonormal basis of  $W$ ?

**Lemma.** Suppose  $\mathbf{A}$  is an  $n \times k$  matrix whose columns are linearly independent. Then  $\mathbf{A}^T \mathbf{A}$  is invertible.

**Theorem.** Let  $\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$  be any basis for a subspace  $W$  of  $\mathbb{R}^n$ . Form the  $n \times k$  matrix

$$\mathbf{A} = \left[ \begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_k \end{array} \right].$$

Then the projection matrix for  $W$  is  $\mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ .

Note that any projection matrix  $\mathbf{P}$  satisfies the two properties

1.  $\mathbf{P}^2 = \mathbf{P}$ , and
2.  $\mathbf{P}$  is symmetric.

It is also true that any matrix that satisfies these two properties is the projection matrix for some subspace of  $\mathbb{R}^n$ .