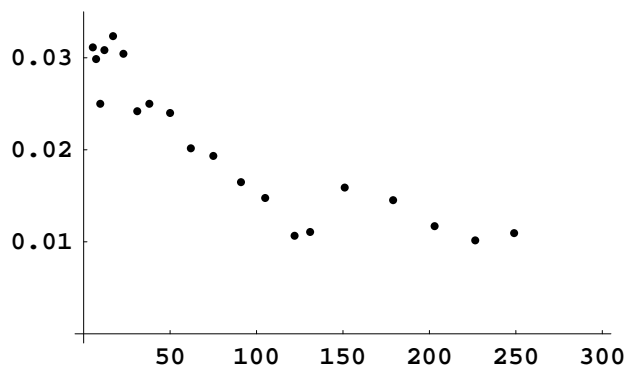


A more realistic example of least squares approximation

Example. Here are relative growth rates for the U.S. population from 1800 to 1990:

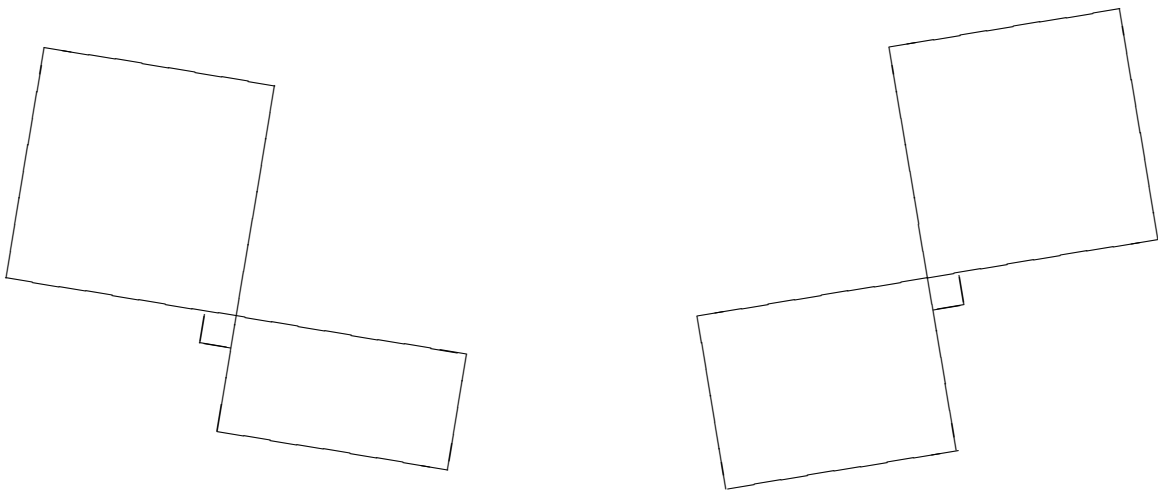
Year	U.S. Population	Rel Growth Rate
1800	5.3	0.03113
1810	7.2	0.02986
1820	9.6	0.02500
1830	12	0.03083
1840	17	0.03235
1850	23	0.03043
1860	31	0.02419
1870	38	0.02500
1880	50	0.02400
1890	62	0.02016
1900	75	0.01933
1910	91	0.01648
1920	105	0.01476
1930	122	0.01066
1940	131	0.01107
1950	151	0.01589
1960	179	0.01453
1970	203	0.01170
1980	226	0.01015
1990	249	0.01094

Here's a graph of these relative growth rates versus population:



Solution sets of linear equations viewed in terms of Strang's figure

There is a version of Strang's figure for systems of linear equations.



Symmetric matrices

Symmetric matrices arise frequently in applications. Moreover, they have a particularly nice structure that can often be used to solve the problem at hand. Today we discuss that structure.

Definition. A matrix \mathbf{A} is symmetric if $\mathbf{A}^T = \mathbf{A}$.

Example. Consider

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

With the aid of *Mathematica*, we see that \mathbf{A} has three distinct real eigenvalues, $\lambda = 7$, $\lambda = 4$, and $\lambda = 1$. We also have three eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

We can diagonalize \mathbf{A} using

Theorem. Let \mathbf{v}_1 and \mathbf{v}_2 be eigenvectors associated to distinct eigenvalues of a symmetric matrix \mathbf{A} . Then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Theorem. (Spectral Theorem for symmetric matrices) If \mathbf{A} is an $n \times n$ symmetric matrix, then

1. \mathbf{A} has n real eigenvalues (counted with multiplicity),
2. the geometric multiplicity of each eigenvalue is the same as its algebraic multiplicity, and
3. distinct eigenspaces are mutually orthogonal.

Consequently, any symmetric matrix is orthogonally diagonalizable.

Note: Any matrix \mathbf{A} that is orthogonally diagonalizable is symmetric.

Example. Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}.$$

Once again with the aid of *Mathematica*, we see that \mathbf{A} has two distinct real eigenvalues, $\lambda = 7$ and $\lambda = -2$. We also have three linearly independent eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$