More on symmetric matrices

Theorem. (Spectral Theorem for symmetric matrices) If $\mathbf{A}$ is an $n \times n$ symmetric matrix, then

1. $\mathbf{A}$ has n real eigenvalues (counted with multiplicity),
2. the geometric multiplicity of each eigenvalue is the same as its algebraic multiplicity, and
3. distinct eigenspaces are mutually orthogonal.

Consequently, any symmetric matrix is orthogonally diagonalizable.

Note: Any matrix $\mathbf{A}$ that is orthogonally diagonalizable is symmetric.

Example. Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}.$$

Once again with the aid of *Mathematica*, we see that $\mathbf{A}$ has two distinct real eigenvalues, $\lambda = 7$ and $\lambda = -2$. We also have three linearly independent eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Suppose that we have an orthogonal diagonalization of the form $\mathbf{D} = \mathbf{P}^T\mathbf{A}\mathbf{P}$. Then

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$$

yields the *spectral decomposition* of $\mathbf{A}$.

Lemma. Let

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \quad\dots\quad & \mathbf{a}_n \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \quad\dots\quad & \mathbf{b}_n \end{array}\right]$$

be two $n \times n$ matrices. Then $\mathbf{A}\mathbf{B}^T = \mathbf{a}_1\mathbf{b}_1^T + \mathbf{a}_2\mathbf{b}_2^T + \ldots + \mathbf{a}_n\mathbf{b}_n^T$.

Now apply the lemma to the product $\mathbf{PDP}^T$ where $\mathbf{P}$ is the orthogonal matrix

$$\mathbf{P} = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \quad \ldots \quad & \mathbf{u}_n \end{array} \right].$$

Example. Let's determine the spectral decomposition for the first example

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

We use the orthonormal basis of eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$