

The span of a set of vectors in  $\mathbb{R}^n$

**Definition.** Given vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  in  $\mathbb{R}^n$  and some choice of real numbers  $r_1, r_2, \dots, r_p$ , then the vector

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_p\mathbf{v}_p$$

is said to be a linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ . The numbers  $r_1, r_2, \dots, r_p$  are called the weights of the linear combination.

**Definition.** Suppose that  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  are vectors in  $\mathbb{R}^n$ . The set of all possible linear combinations of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  is called the

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}.$$

Note:

1. Every scalar multiple of each  $\mathbf{v}_k$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ .
2. The zero vector is always in the span of any set of vectors.
3.  $\text{Span}\{\mathbf{v}_1\}$  is the set of all scalar multiples of  $\mathbf{v}_1$ .

**Example.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

What vectors are in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ ?

**Example.** Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}.$$

For what values of  $x_3$  is the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ x_3 \end{bmatrix}$$

in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ ? What does this result mean geometrically?

The matrix-vector product  $\mathbf{Ax}$

Let  $\mathbf{A}$  be an  $m \times n$  matrix and  $\mathbf{x}$  be a vector in  $\mathbb{R}^n$ . We can define the product  $\mathbf{Ax}$  as a linear combination of the vectors that come from the columns of  $\mathbf{A}$ .

**Definition.** Let  $\mathbf{A}$  be an  $m \times n$  matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[ \begin{array}{c|c|c|c} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{array} \right],$$

where  $\mathbf{A}_k$  is the  $k$ th column of  $\mathbf{A}$ . Given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

in  $\mathbb{R}^n$ , we define the matrix-vector product  $\mathbf{Ax}$  to be the linear combination

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n.$$

Note that  $\mathbf{Ax}$  is a vector in  $\mathbb{R}^m$ .

**Example.**

$$\begin{aligned} \begin{bmatrix} 3 & -8 \\ -1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} &= -4 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ 5 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} (-4)(3) + (2)(-8) \\ (-4)(-1) + (2)(5) \\ (-4)(2) + (2)(3) \end{bmatrix} \\ &= \begin{bmatrix} -28 \\ 14 \\ -2 \end{bmatrix} \end{aligned}$$

**Remark.** Given an  $m \times n$  matrix  $\mathbf{A}$  and  $\mathbf{x} \in \mathbb{R}^n$ , then the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\left[ \mathbf{A}_1 \mid \mathbf{A}_2 \mid \dots \mid \mathbf{A}_n \mid \mathbf{b} \right] = \left[ \mathbf{A} \mid \mathbf{b} \right].$$

**Theorem.** Let  $\mathbf{A}$  be an  $m \times n$  matrix. Then the following three statements are equivalent:

1. For each  $\mathbf{b}$  in  $\mathbb{R}^m$ , the equation  $\mathbf{Ax} = \mathbf{b}$  has at least one solution.
2. The columns of  $\mathbf{A}$  span  $\mathbb{R}^m$ .
3. The matrix  $\mathbf{A}$  has a pivot position in every row.

**Warning:** In this theorem,  $\mathbf{A}$  is a *coefficient* matrix. The three statements are not equivalent if  $\mathbf{A}$  is an augmented matrix.

**Observation.** Note that the  $k$ th entry in  $\mathbf{Ax}$  is

$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n.$$

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n$$

is called the **dot product** of  $[a_{k1} \ a_{k2} \ \dots \ a_{kn}]$  and the vector  $\mathbf{x}$ .

**Theorem.** Let  $\mathbf{A}$  be an  $m \times n$  matrix. Then the matrix-vector product  $\mathbf{Ax}$  is “linear” in  $\mathbf{x}$ . That is,

1.  $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}$  for all  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$ , and
2.  $\mathbf{A}(c\mathbf{u}) = c\mathbf{Au}$  for all  $\mathbf{u}$  in  $\mathbb{R}^n$  and all  $c$  in  $\mathbb{R}$ .