More on the matrix-vector product $\mathbf{Ax}$

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution.
2. The columns of $\mathbf{A}$ span $\mathbb{R}^m$.
3. The matrix $\mathbf{A}$ has a pivot position in every row.

Warning: In this theorem, $\mathbf{A}$ is a *coefficient* matrix. The three statements are not equivalent if $\mathbf{A}$ is an augmented matrix.

Observation. Note that the kth entry in $\mathbf{Ax}$ is

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n.$$

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n$$

is called the **dot product** of $[\,a_{k1} \quad a_{k2} \quad \ldots \quad a_{kn}\,]$ and the vector $\mathbf{x}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. Then the matrix-vector product $\mathbf{Ax}$ is "linear" in $\mathbf{x}$. That is,

1. $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and
2. $\mathbf{A}(c\mathbf{u}) = c\mathbf{Au}$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and all c in $\mathbb{R}$.

Solution sets of systems of linear equations

Definition. Consider a linear system $\mathbf{Ax} = \mathbf{b}$. We say that it is *homogeneous* if $\mathbf{b} = \mathbf{0}$ and *nonhomogeneous* otherwise.

The homogeneous case $\mathbf{Ax} = \mathbf{0}$

Observation. Note that every homogeneous system is consistent. The solution $\mathbf{x} = \mathbf{0}$ is called the *trivial* solution. All other solutions are said to be nontrivial.

Theorem. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$, then any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is also a solution.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Express the solution set for $\mathbf{Ax} = \mathbf{0}$ as a span. (Note that $\mathbf{A}$ is a coefficient matrix, not an augmented matrix.)

The nonhomogeneous case $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$

Theorem. Let $\mathbf{p}$ be one solution to the nonhomogeneous equation and let S be the set of all solutions to the associated homogeneous equation

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

Then the solution set of $\mathbf{A}\mathbf{x} = \mathbf{b}$ consists of all vectors in the set

$$\mathbf{p} + S.$$

Example. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ whose augmented matrix is

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 9 \end{array}\right].$$