More on linear transformations

Basic facts about linear transformations T

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(r_1\mathbf{v}_1 + \ldots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + \ldots + r_kT(\mathbf{v}_k)$

Last class we discussed the following list of functions. (Some are linear, but others are not.)

Examples: functions $f : \mathbb{R} \to \mathbb{R}$

1. $f_1(x) = 2x$
2. $f_2(x) = 2x + 1$
3. $f_3(x) = x^2$
4. $f_4(x) = \cos x$

Examples: functions $g : \mathbb{R}^2 \to \mathbb{R}^2$

1. $g_1(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$
2. $g_2(x_1, x_2) = (\cos(x_1 + x_2), x_1 + x_2^2)$

Examples: functions h defined on $\mathbb{R}^3$

1. $h_1(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_2 + x_3)$
2. $h_2(x_1, x_2, x_3) = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

One way to show that a transformation is linear is to verify the two conditions of linearity directly, but there is an easier way to see that these transformations are linear.

Important class of examples: Given an $m \times n$ matrix $\mathbf{A}$, then we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by the equation

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

We know that T is a linear transformation because the matrix-vector product satisfies the necessary conditions.

Example. Let

$$\mathbf{G} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Then

$$\mathbf{G}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

On the course web page, there is a link to a java program called `Matrix Machine` written by Hu Hohn. It lets you investigate the mapping properties of various 2 × 2 matrix transformations. Try the following three matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

There are also links to two applets by David Austin at Grand Valley State University. These two applets are particularly useful when you want to understand the mapping properties of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Theorem. Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $m \times n$ matrix. This matrix $\mathbf{A}$ is called the *standard matrix representation* of T.

Why? Let's make two observations using the "standard basis" of $\mathbb{R}^n$.

Definition. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \ldots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

1. If we know the images $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ for all of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then we can calculate $T(\mathbf{x})$ for any $\mathbf{x}$ in $\mathbb{R}^n$.

2. If

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \quad\ldots\quad & T(\mathbf{e}_n) \end{array}\right],$$

then

$$\mathbf{A}\mathbf{x} = x_1 T(\mathbf{e}_1) + \ldots + x_n T(\mathbf{e}_n).$$

Example. What is the matrix representation of the linear transformation that rotates $\mathbb{R}^2$ by 45° ($\pi/4$ radians) around the origin?

Examples of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$

1. rotations
2. reflections
3. contractions and expansions
4. shears
5. projections