

More on linear transformations

Examples of linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

1. rotations

2. reflections

3. contractions and expansions

4. shears

5. projections

Definitions. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a transformation.

1. We say that T maps \mathbb{R}^n onto \mathbb{R}^m if each \mathbf{b} in \mathbb{R}^m is the image of at least one \mathbf{x} in \mathbb{R}^n .
2. We say that T is one-to-one if each \mathbf{b} in \mathbb{R}^m is the image of at most one \mathbf{x} in \mathbb{R}^n .

Examples. Let

$$\mathbf{P}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Theorem. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the only solution to $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ is the trivial solution.

Theorem. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let \mathbf{A} be its standard matrix representation. Then

1. T maps \mathbb{R}^n onto \mathbb{R}^m if and only if the columns of \mathbf{A} span \mathbb{R}^m , and
2. T is one-to-one if and only if the columns of \mathbf{A} are linearly independent.