Two theorems on mapping properties of linear transformations

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the only solution to $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ is the trivial solution.

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\mathbf{A}$ be its standard matrix representation. Then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $\mathbf{A}$ span $\mathbb{R}^m$, and
2. T is one-to-one if and only if the columns of $\mathbf{A}$ are linearly independent.

Matrix algebra

Matrices have an unusual algebraic structure associated to them.

Matrix addition and scalar multiplication should remind you of vector addition and scalar multiplication in $\mathbb{R}^n$.

Definitions.

1. Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times n$ matrices. Then the sum of $\mathbf{A}$ and $\mathbf{B}$ is the $m \times n$ matrix $\mathbf{C}$ where
$$c_{ij} = a_{ij} + b_{ij}.$$
2. Let $\mathbf{A}$ be an $m \times n$ matrix and r be a real number. Then the scalar multiple $r\mathbf{A}$ is the matrix whose ij-entry is ra_{ij}. In other words, every entry of $\mathbf{A}$ is multiplied by r.

Theorem 1. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $m \times n$ matrices and let r and s be real numbers. Then

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$
4. $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
5. $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$
6. $r(s\mathbf{A}) = (rs)\mathbf{A}$

Matrix multiplication is more subtle than addition or scalar multiplication, and there are two ways to define it. One way is algebraic. The other is geometric.

Here is the geometric definition:

Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{B} : \mathbb{R}^p \to \mathbb{R}^n$ be linear transformations. What is the matrix representation of the composition $\mathbf{A} \circ \mathbf{B}$? (Note that this composition has $\mathbf{B}$ as the first transformation.)

Definition. The product $\mathbf{C}$ of $\mathbf{A}$ and $\mathbf{B}$ is the matrix

$$\mathbf{C} = \left[\begin{array}{c|c|c|c} \mathbf{AB}_1 & \mathbf{AB}_2 & \quad \dots \quad & \mathbf{AB}_p \end{array} \right].$$

Let's check the sizes of all of the matrices involved.

Example. Let $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 45° and let $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation determined by the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

What is the matrix representation for the transformation $\mathbf{A} \circ \mathbf{B}$?

Row-column dot product definition: The columns of $\mathbf{AB}$ are linear combinations of the columns of $\mathbf{A}$. In fact, consider the jth column of $\mathbf{AB}$.

Row-column rule: $(\mathbf{AB})_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$

Definition. The $n \times n$ (square) matrix

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix}$$

with 1's down the diagonal and 0's everywhere else is called the $n \times n$ *identity matrix.*

Theorem 2. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ and $\mathbf{C}$ be matrices of appropriate sizes. Then

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
4. $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B} = \mathbf{A}(r\mathbf{B})$ for any scalar r
5. $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$

Three warnings.

1. $\mathbf{AB}$ does not always equal $\mathbf{BA}$.
2. $\mathbf{AB} = \mathbf{AC}$ does not necessarily imply that $\mathbf{B} = \mathbf{C}$.
3. $\mathbf{AB} = \mathbf{0}$ does not necessarily imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.