Properties of the Determinant

In order to gain some insight into how we will compute determinants in general, let's calculate the determinants of all elementary 3×3 matrices.

Theorem. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Then

1. The matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.
2. $\det \mathbf{A}^T = \det \mathbf{A}$
3. $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$

Given the fact that $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$, we can consider the determinant of the product

$$\mathbf{EA}$$

where $\mathbf{E}$ is an elementary matrix.

Row operations and the determinant:

1. Suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying exactly one row replacement row operation, then
$$\det \mathbf{B} =$$

2. Suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying exactly one row swap row operation, then
$$\det \mathbf{B} =$$

3. Suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying exactly one row scaling row operation, then
$$\det \mathbf{B} =$$

Corollary. If $\mathbf{A}$ has two identical rows, then

$$\det \mathbf{A} = 0.$$

Proof of the fact that doing a row replacement row operation does not change the determinant: Suppose that

$$\mathbf{B} = \left[\begin{array}{c} R_1 \\ \hline \vdots \\ \hline R_i + \alpha R_j \\ \hline \vdots \\ \hline R_n \end{array}\right]$$

where R_1, R_2, …, R_n represent the rows of $\mathbf{A}$.

Example. Consider the 4×4 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 4 & 14 \\ 4 & 3 & 1 & 2 \\ -1 & 8 & 6 & 2 \\ 2 & -2 & 4 & -3 \end{bmatrix}$$

Let's calculate the determinant of $\mathbf{A}$ using row operations.

Some practice with the properties of determinants:

Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 matrices with $\det \mathbf{A} = 3$ and $\det \mathbf{B} = -2$. Compute:

1. $\det \mathbf{AB}$

2. $\det \mathbf{B}^5$

3. $\det 2\mathbf{A}$

4. $\det \mathbf{A}^{\mathrm{T}}\mathbf{A}$

5. $\det \mathbf{B}^{-1}\mathbf{AB}$