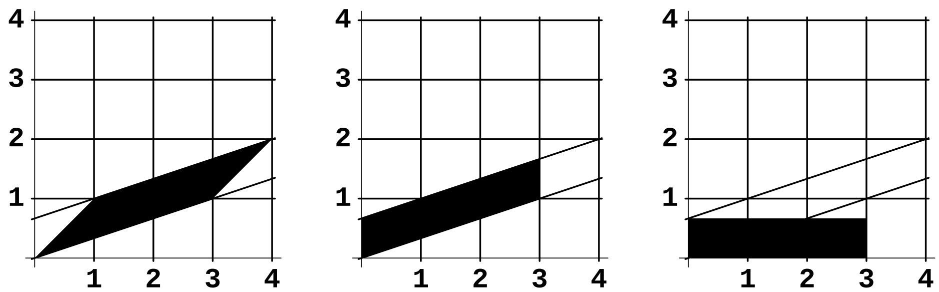
More on the geometry of the determinant

Example. Let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Yesterday we calculated the area of the parallelogram formed by **u** and **v** in an unusual way.



We can represent these two geometric manipulations as column operations on a matrix.

Summary: 2×2 Matrices and Area in $\mathbb{R}^2$

Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

then the area of the parallelogram formed by **u** and **v** is

$$\left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|.$$

Volume in $\mathbb{R}^3$

The same basic geometric argument that works in $\mathbb{R}^2$ also works in $\mathbb{R}^3$. Given three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$, consider the parallelopiped P that they determine.

How do column operations change the parallelopiped, and what do they do to the corresponding volumes?

Example. Consider the parallelopiped generated by

$$\mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix},$$

Summary: 3×3 Matrices and Volume in $\mathbb{R}^3$

Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then the volume of the parallelopiped formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is

$$\left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|.$$

Determinants and linear transformations

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a square matrix $\mathbf{A}$ such that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

What is the significance of $\det \mathbf{A}$ in this situation?

Consider the case where $n = 2$ and start with a parallelogram P determined by two vectors $\mathbf{u}$ and $\mathbf{v}$.

Summary: Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, then $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some 2×2 matrix $\mathbf{A}$. If P is a parallelogram determined by the two vectors $\mathbf{u}$ and $\mathbf{v}$, then $T(P)$ is also a parallelogram, and

$$\text{area of } T(P) = (\det A)(\text{area of } P).$$

For $n = 3$, the same conclusion holds if the concept of area is replaced by that of volume.

Also, there is nothing special about parallelograms in this discussion. We could just as well start with a region such as a disk.

For more details, see pp. 208–209 of our text.

If you have studied multivariable calculus, you know that there is a change of variables formula that is used to convert multiple integrals from one set of coordinates to another. That formula involves the determinant of the Jacobian matrix (see Stewart *Calculus: Concepts and Contexts*, Section 12.9). The area conversion formula mentioned here is a special case of that more general formula.