What is a vector space?

Definition. A vector space V is a set of objects that are called vectors along with two operations—vector addition and scalar multiplication. The vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is always defined for any pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, and given any scalar r in $\mathbb{R}$ and any vector $\mathbf{v}$ in V, the scalar multiple $r\mathbf{v}$ is a vector in V. Moreover, these two operations must satisfy the following eight properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For each $\mathbf{u}$, there is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Example 1. The vector space $\mathbb{R}^n$. See p. 32 of our text for a discussion of the properties listed above.

Example 2. The vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

Given two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, the vector sum $f + g$ of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

for all x in $\mathbb{R}$.

Given a function $f : \mathbb{R} \to \mathbb{R}$ and a real number r, the scalar multiple rf is defined by

$$(rf)(x) = r(f(x))$$

for all x in $\mathbb{R}$.

The operations of vector addition and scalar multiplication are illustrated by graphs that are posted on the course web site. In addition, at the same location, the concept of a linear combination of two functions is illustrated by two examples and their graphs.

Example 3. The vector space $M_{m\times n}$ of all $m \times n$ matrices. (We assume that the entries of the matrices are real numbers, but a different vector space is obtained if one allows the entries to be complex numbers.)

The operation of vector addition is the usual operation of addition for two matrices of the same size. The operation of scalar multiplication is the product of a real number and a matrix that was defined a couple of weeks ago.

The operations of vector addition and scalar multiplication and the concept of linear combination of two "vectors" are illustrated by examples that are posted on the course web site.

Example 4. The vector space $\mathbb{P}$ of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$.

Since polynomial functions qualify as functions as in Example 2, we can define vector addition and scalar multiplication just as we did in Example 2. Once again these operations are illustrated by formulas and graphs that are posted on the course web site.

Example 5. The vector space of all discrete-time signals.

Subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S,
2. (closure under vector addition) for each $\mathbf{v}_1$ and $\mathbf{v}_2$ in S, the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S, and
3. (closure under scalar multiplication) for each r in $\mathbb{R}$ and each $\mathbf{v}$ in S, the scalar multiple $r\mathbf{v}$ is in S.

Note. A subspace S of a vector space V is a vector space in its own right.

Example. Consider the line $x_2 = 3x_1$ in the vector space $\mathbb{R}^2$.

Example. Consider the line $x_2 = x_1 + 1$ in the vector space $\mathbb{R}^2$.