

More on subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S ,
2. (closure under vector addition) for each \mathbf{v}_1 and \mathbf{v}_2 in S , the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S , and
3. (closure under scalar multiplication) for each r in \mathbb{R} and each \mathbf{v} in S , the scalar multiple $r\mathbf{v}$ is in S .

Note. A subspace S of a vector space V is a vector space in its own right.

Examples. Last class we saw that the line $x_2 = 3x_1$ is a subspace of the vector space \mathbb{R}^2 .

Also the line $x_2 = x_1 + 1$ is not a subspace of \mathbb{R}^2 .

Example. Let \mathbb{P} represent the vector space of all polynomial functions as discussed last class. Is \mathbb{P} a subspace of the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within \mathbb{P} . Is S a subspace of \mathbb{P} ?

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are vectors in a vector space V , then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a subspace of V .

Example. Let V be the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Which of the following subsets of V are subspaces of V ?

1. The set of all constant functions.
2. The set of all functions f such that $f(2) = 1$.
3. The set of all functions f such that $f(2) = 0$.
4. The set of all polynomials of degree 3.
5. The set of all polynomials whose degree is at most 3.
6. The set of all differentiable functions.

Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix \mathbf{A} . Let $\mathbf{c}_1, \dots, \mathbf{c}_n$ represent the columns of \mathbf{A} . That is,

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{array} \right].$$

These column vectors are vectors in \mathbb{R}^m .

Let $\mathbf{r}_1, \dots, \mathbf{r}_m$ represent the rows of \mathbf{A} . That is,

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{array} \right].$$

These row vectors are vectors in \mathbb{R}^n .

The column space of \mathbf{A} . The column space of \mathbf{A} is the span of the columns of \mathbf{A} . We write

$$\text{Col } \mathbf{A} = \text{Span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}.$$

The row space of \mathbf{A} . The row space of \mathbf{A} is the span of the rows of \mathbf{A} . We write

$$\text{Row } \mathbf{A} = \text{Span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\}.$$

The null space of \mathbf{A} . The null space of \mathbf{A} is the set of all vectors \mathbf{x} in \mathbb{R}^n such that

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

The null space of \mathbf{A} is denoted by $\text{Nul } \mathbf{A}$.

Theorem. Let \mathbf{A} be an $m \times n$ matrix. The column space of \mathbf{A} is a subspace of \mathbb{R}^m , and the null space and the row space of \mathbf{A} are subspaces of \mathbb{R}^n .

Application. Any plane through the origin in \mathbb{R}^3 is a subspace of \mathbb{R}^3 .