More on the kernel and range of a linear transformation

For a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ determined by the matrix $\mathbf{A}$, its range is Col $\mathbf{A}$, and its kernel is Nul $\mathbf{A}$.

Example. What are the kernel and range of the transformation $p : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}?$$

(This example was first introduced on September 22.)

Example. Let V_1 be the vector space of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and let V_2 be the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. The operation of differentiation is a linear transformation from V_1 to V_2. That is, the transformation $D : V_1 \to V_2$ given by $D(f) = f'$ is a linear transformation.

Example. What are the kernel and the range of the differentiation transformation D mentioned above?

If we are careful, we can also use integration to define a linear transformation.

Example. Given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ in $\mathbb{P}_n$ (the vector space of all polynomial functions of degree at most n), we can define

$$I(p) = \int_0^x p(t)\,dt = a_n \frac{x^{n+1}}{n+1} + a_{n-1}\frac{x^n}{n} + \ldots + a_0 x.$$

The map $I : \mathbb{P}_n \to \mathbb{P}_{n+1}$ is a linear transformation. What are its kernel and range?

Bases for vector spaces and subspaces

Given a vector space or subspace V, we often find it convenient to express it as the span of a few vectors. A basis for V is a spanning set that contains as few vectors as possible.

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a *basis* for V if

1. it is linearly independent, and
2. it spans V.

Example. The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

Example. The two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^2$.

Example. The set $\{x^3, x^2, x, 1\}$ is a basis of $\mathbb{P}_3$.

Example. The set $\{x^3, x^3 + x^2, x, 1\}$ is another basis of $\mathbb{P}_3$.

We need ways of determining bases of vector spaces and their subspaces. The "casting-out procedure" produces a basis from a spanning set.

The casting-out procedure

Given a vector subspace S spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can obtain a basis B for S by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
   (a) (cast out) $B_i = B_{i-1}$ if $\mathbf{v}_i$ is in $\operatorname{Span} B_{i-1}$, or
   (b) (keep) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if $\mathbf{v}_i$ is not in $\operatorname{Span} B_{i-1}$.

Then the final result B_k is a basis B for S.

Example. Let's apply the casting-out procedure to the set $\{x^3+1, x, x^2, x^2-x, 4, x^3\}$ of polynomials in $\mathbb{P}_3$.

Theorem. (similar to The Spanning Set Theorem, Lay, p. 239) Let $S = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then the final result B_k of the casting-out procedure applied to $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for S.

The proof of the casting-out procedure is posted on the web site, and we will not discuss it in class. However, to understand the proof of the theorem, it is helpful to consider the example above along with the sets B_1, B_2, ..., B_6. We get

$$B_1 = \left\{x^3+1\right\}$$

$$B_2 = \left\{x^3+1, x\right\}$$

$$B_3 = \left\{x^3+1, x, x^2\right\}$$

$$B_4 = B_3$$

$$B_5 = \left\{x^3+1, x, x^2, 4\right\}$$

$$B_6 = B_5.$$