How do we find the eigenvalues?

Note: The number λ is an eigenvalue for the matrix $\mathbf{A}$ if and only if the homogeneous system

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

By the Invertible Matrix Theorem,

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if the matrix $(\mathbf{A} - \lambda \mathbf{I})$ is **not** invertible.

The number λ is an eigenvalue for the matrix $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Theorem. If a matrix is upper/lower triangular, then its eigenvalues are the entries along the main diagonal.

Now I want to use the computer to examine the characteristic polynomial for various matrices.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & -2 \\ -2 & 2 & -2 & 0 \\ -2 & 5 & 4 & -4 \\ 3 & 6 & -6 & -6 \end{bmatrix}.$$

In theory, any polynomial $p(\lambda)$ can be factored into irreducible linear and quadratic factors using real numbers. For example, consider the polynomial

$$p(\lambda) = \lambda^9 + 8\lambda^8 + 36\lambda^7 + 94\lambda^6 + 143\lambda^5 + 98\lambda^4 - 48\lambda^3 - 160\lambda^2 - 132\lambda - 40.$$

This polynomial factors into

$$p(\lambda) = (\lambda^2 + 2\lambda + 2)^2(\lambda^2 + 3\lambda + 10)(\lambda + 1)^2(\lambda - 1).$$

The **algebraic multiplicity** of an eigenvalue λ_0 is the number of times that the factor $(\lambda - \lambda_0)$ appears in the factorization of the characteristic polynomial $p(\lambda)$. The **geometric multiplicity** of λ_0 is the dimension of its eigenspace.

Theorem. The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.

Example. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$