

More on orthogonal projection

Recall the Orthogonal Decomposition Theorem from last class.

**Theorem.** (Orthogonal Decomposition Theorem)

1. Each vector  $\mathbf{v}$  in  $\mathbb{R}^n$  can be written uniquely as

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp,$$

where  $\mathbf{w}$  is in  $W$  and  $\mathbf{w}^\perp$  is in  $W^\perp$ .

2. Given an orthogonal basis  $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$  of  $W$ , then

$$\text{proj}_W \mathbf{v} \equiv \mathbf{w} = \left( \frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 + \dots + \left( \frac{\mathbf{v} \cdot \mathbf{w}_k}{\mathbf{w}_k \cdot \mathbf{w}_k} \right) \mathbf{w}_k$$

and  $\mathbf{w}^\perp = \mathbf{v} - \mathbf{w}$ .

Note: Since the two vectors  $\mathbf{w}$  and  $\mathbf{w}^\perp$  are unique, they do not depend on the orthogonal basis of  $W$  that we use to compute them.



Important consequence: If we want to find the distance of a vector  $\mathbf{v}$  to a subspace  $W$ , then we compute

$$\|\mathbf{w}^\perp\| = \|\mathbf{v} - \text{proj}_W \mathbf{v}\|.$$

**Example.** Find the point closest to

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

in the subspace  $W$  spanned by the two vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

**Theorem.** If  $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$  is an orthonormal basis for a subspace  $W$ , then

$$\text{proj}_W \mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1) \mathbf{u}_1 + \dots + (\mathbf{v} \cdot \mathbf{u}_k) \mathbf{u}_k.$$

If

$$\mathbf{U} = \left[ \begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k \end{array} \right],$$

then  $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T \mathbf{v}$ .

**Example.** Let's repeat the previous calculation using the projection matrix. Let

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{4}{\sqrt{26}} \\ -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{26}} \\ -\frac{1}{\sqrt{10}} & 0 \\ \frac{2}{\sqrt{10}} & \frac{3}{\sqrt{26}} \end{bmatrix}.$$

Then

$$\mathbf{U}\mathbf{U}^T = \begin{bmatrix} \frac{93}{130} & \frac{23}{65} & -\frac{1}{10} & -\frac{17}{65} \\ \frac{23}{65} & \frac{57}{130} & \frac{1}{5} & -\frac{37}{130} \\ -\frac{1}{10} & \frac{1}{5} & \frac{1}{10} & -\frac{1}{5} \\ -\frac{17}{65} & -\frac{37}{130} & -\frac{1}{5} & \frac{97}{130} \end{bmatrix}.$$

## The Gram-Schmidt Process

This procedure produces an orthogonal (or orthonormal) basis from a basis  $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$  of a subspace  $W$ . It is an inductive procedure.

We work with the subspaces

$$S_l = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_l\}.$$

The orthogonal basis for  $W$  based on this procedure applied to this basis is denoted  $\{\mathbf{v}_1, \dots, \mathbf{v}_l\}$ .

1. Let  $\mathbf{v}_1 = \mathbf{x}_1$ .
2. Let  $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ .
3. Let  $\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2$ .

etc.

**Example.** Apply the Gram-Schmidt process to the basis

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$