The rank of a matrix

Recall that the row space of an $m \times n$ matrix $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$.

Theorem. If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then Row $\mathbf{A}$ = Row $\mathbf{B}$.

How do we find a basis for Row $\mathbf{A}$?

Example. Last class we saw that the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent.

For an $m \times n$ matrix $\mathbf{A}$, we saw that

$$\dim(\text{Col } \mathbf{A}) + \dim(\text{Nul } \mathbf{A}) = n.$$

How is the dimension of Row $\mathbf{A}$ related to these numbers?

Definition. The rank of an $m \times n$ matrix $\mathbf{A}$ is the dimension of its column space. This dimension also equals the dimension of Row $\mathbf{A}$.

Example. Suppose that a homogeneous linear system of 10 equations in 6 unknowns has two linearly independent solutions and all other solutions are linear combinations of these. Can the solution set be described with fewer equations? If so, how many?

More equivalent conditions can be added to the Invertible Matrix Theorem.

Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following seven statements are equivalent.

1. The matrix $\mathbf{A}$ is invertible.
2. The columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$.
3. $\text{Col } \mathbf{A} = \mathbb{R}^n$
4. $\dim(\text{Col } \mathbf{A}) = n$
5. $\text{rank}(\mathbf{A}) = n$
6. $\text{Nul } \mathbf{A} = \{\mathbf{0}\}$
7. $\dim(\text{Nul } \mathbf{A}) = 0$

Since $\mathbf{A}$ is invertible if and only if $\mathbf{A}^T$ is invertible, any statement regarding the columns of $\mathbf{A}$ in the Invertible Matrix Theorem can be replaced by a statement regarding the rows of $\mathbf{A}$.

Rank and linear transformations

Consider a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{A}$ be its standard matrix representation. How is the rank of $\mathbf{A}$ related to the mapping properties of L?

Note: The range of L is the column space of $\mathbf{A}$ in $\mathbb{R}^m$. Consequently, $\text{rank}(\mathbf{A})$ is the dimension of the range of L. This number is often called the rank of L.

Example. Consider a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$. What is its rank?

Such a matrix is said to be of "full rank."

Example. What is the rank of the linear transformation $P : \mathbb{R}^2 \to \mathbb{R}^2$ that projects $\mathbb{R}^2$ onto the line $x_2 = -x_1$?

Example. Consider the linear transformation $p : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix

$$\mathbf{P} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(This example was first introduced on September 22 and discussed again on October 24.) What is its rank?

Theorem. Let $\mathbf{A}$ and $\mathbf{B}$ be transformations/matrices such that $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$. Then

$$\operatorname{rank}(\mathbf{AB}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$