Eigenvectors and eigenvalues

Eigenvalues and eigenvectors are special numbers and vectors associated to certain matrices. They are useful in many applications. In particular, we will use them to help us understand how a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ transforms $\mathbb{R}^n$.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} \frac{31}{45} & \frac{19}{45} \\ -\frac{19}{90} & \frac{119}{90} \end{bmatrix}.$$

There is a Quicktime movie on the web site that illustrates how $\mathbf{A}$ and its powers $\mathbf{A}^2, \ldots, \mathbf{A}^7$ transform the unit square in $\mathbb{R}^2$.

Definition. Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathbf{x}$ is a nonzero vector such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

for some scalar λ, then λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is an eigenvector associated to the eigenvalue λ.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

There is a link on the web page to a java applet called `Eigen Engine`. It helps us visualize a few special cases.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}.$$

Basic facts about eigenvectors

1. Any nonzero scalar multiple of an eigenvector is another eigenvector associated to the same eigenvalue.

2. The equation $\mathbf{Ax} = \lambda\mathbf{x}$ can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}.$$

   If λ is an eigenvalue for $\mathbf{A}$, then Nul $(\mathbf{A} - \lambda\mathbf{I})$ is the eigenspace associated to λ.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The numbers $\lambda = 1$ and $\lambda = 2$ are eigenvalues of $\mathbf{A}$. What are the corresponding eigenspaces?

The $\lambda = 1$ eigenspace:

The $\lambda = 2$ eigenspace:

Theorem. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

How do we find the eigenvalues?

Note: The number λ is an eigenvalue for the matrix $\mathbf{A}$ if and only if the homogeneous system

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

By the Invertible Matrix Theorem,

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if the matrix $(\mathbf{A} - \lambda \mathbf{I})$ is **not** invertible.

The number λ is an eigenvalue for the matrix $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$