More on the Gram-Schmidt Process

Recall that this procedure produces an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ for a subspace W from an arbitrary basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$.
3. Let $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \dfrac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$.

etc.

Example. Apply the Gram-Schmidt process to the basis

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

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Projection matrices

We discussed projection matrices briefly last class. In particular, we discussed the following theorem.

Theorem. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$\mathbf{U} = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \quad \cdots \quad & \mathbf{u}_k \end{array} \right].$$

Then $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T \mathbf{v}$.

The matrix $\mathbf{P} = \mathbf{U}\mathbf{U}^T$ is called the *projection matrix* for the subspace W. It does not depend on the choice of orthonormal basis of W.

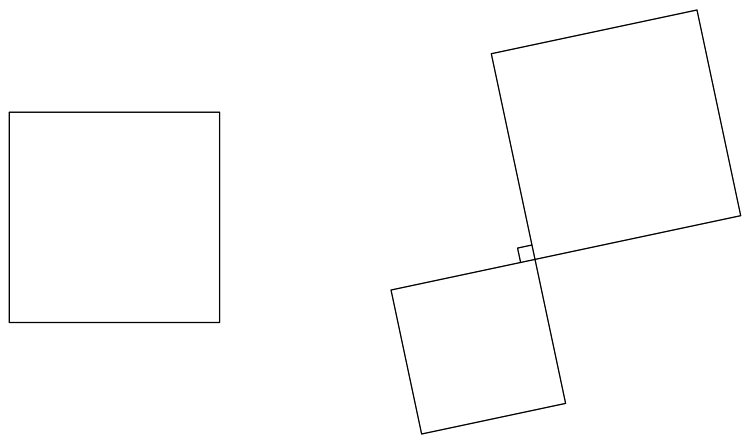
Example. Let's compute the projection matrix $\mathbf{P}$ for orthogonal projection onto the plane $x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$.

What are the eigenvalues and eigenspaces of $\mathbf{P}$? (No computation required)

What if we do not start with an orthonormal basis of W?

Lemma. Suppose $\mathbf{A}$ is an $n \times k$ matrix whose columns are linearly independent. Then $\mathbf{A}^T\mathbf{A}$ is invertible.

Here's a diagram due to Gilbert Strang (MIT) that helps us understand why this lemma is true.



Theorem. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be any basis for a subspace W of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_k \end{array} \right].$$

Then the projection matrix for W is $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.