

The span of a set of vectors in \mathbb{R}^n

Definition. Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and some choice of real numbers r_1, r_2, \dots, r_p , then the vector

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_p\mathbf{v}_p$$

is said to be a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. The numbers r_1, r_2, \dots, r_p are called the weights of the linear combination.

Definition. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are vectors in \mathbb{R}^n . The set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is called the

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}.$$

Note:

1. Every scalar multiple of each \mathbf{v}_k is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$.
2. The zero vector is always in the span of any set of vectors.
3. $\text{Span}\{\mathbf{v}_1\}$ is the set of all scalar multiples of \mathbf{v}_1 .

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

What vectors are in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}.$$

For what values of x_3 is the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ x_3 \end{bmatrix}$$

in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? What does this result mean geometrically?

The matrix-vector product \mathbf{Ax}

Let \mathbf{A} be an $m \times n$ matrix and \mathbf{x} be a vector in \mathbb{R}^n . We can define the product \mathbf{Ax} as a linear combination of the vectors that come from the columns of \mathbf{A} .

Definition. Let \mathbf{A} be an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{array} \right],$$

where \mathbf{A}_k is the k th column of \mathbf{A} . Given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

in \mathbb{R}^n , we define the matrix-vector product \mathbf{Ax} to be the linear combination

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n.$$

Note that \mathbf{Ax} is a vector in \mathbb{R}^m .

Example.

$$\begin{aligned} \begin{bmatrix} 3 & -8 \\ -1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} &= -4 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ 5 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} (-4)(3) + (2)(-8) \\ (-4)(-1) + (2)(5) \\ (-4)(2) + (2)(3) \end{bmatrix} \\ &= \begin{bmatrix} -28 \\ 14 \\ -2 \end{bmatrix} \end{aligned}$$

Remark. Given an $m \times n$ matrix \mathbf{A} and $\mathbf{x} \in \mathbb{R}^n$, then the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\left[\mathbf{A}_1 \mid \mathbf{A}_2 \mid \dots \mid \mathbf{A}_n \mid \mathbf{b} \right] = \left[\mathbf{A} \mid \mathbf{b} \right].$$

Theorem. Let \mathbf{A} be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each \mathbf{b} in \mathbb{R}^m , the equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution.
2. The columns of \mathbf{A} span \mathbb{R}^m .
3. The matrix \mathbf{A} has a pivot position in every row.

Warning: In this theorem, \mathbf{A} is a *coefficient* matrix. The three statements are not equivalent if \mathbf{A} is an augmented matrix.