

Linear independence

Definition. A (linear) dependence relation among a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an equation of the form

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some vector $\mathbf{v}_i \in \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

Example.

$$2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition. If there exists a dependence relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0}$$

among a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, then we say that the set is *linearly dependent*. A set is *linearly independent* if it is not linearly dependent.

Matrix characterization

A dependence relation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0}$ can be rewritten as the matrix equation $\mathbf{A}\mathbf{r} = \mathbf{0}$ where

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{array} \right] \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}.$$

Therefore, a dependence relation among the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the same as a nontrivial solution to $\mathbf{A}\mathbf{r} = \mathbf{0}$.

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We can determine that these three vectors are linearly independent by considering the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Example. Which of the following sets of vectors in \mathbb{R}^3 are linearly independent? Why?

1. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ \pi \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$

Theorem. A nonzero set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors is linearly dependent if and only if, for some index j , the vector \mathbf{v}_j is a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Theorem. If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set of vectors in \mathbb{R}^n , then $k \leq n$.

Example. We know that the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$$

is linearly dependent. What are all possible dependence relations among this set of vectors?