1. (12 points) Let

$$\mathbf{A} = \begin{bmatrix} -3 & 2 & -2 & 1 \\ 3 & 2 & -5 & 0 \\ -2 & -4 & 1 & -5 \\ 0 & 0 & 1 & -7 \end{bmatrix}.$$

Compute a basis for the $\lambda = -4$ eigenspace of $\mathbf{A}$.

2. (16 points) Consider the system

$$\begin{aligned} x_1 + 3x_4 &= 4 \\ x_2 + x_3 - 3x_4 &= 2 \\ -x_1 - 3x_4 + x_5 &= -1 \\ 3x_1 + 9x_4 - 2x_5 &= 6 \end{aligned}$$

of four equations in five unknowns.

(a) Express its solution set in parametric vector form.

(There is extra space for your answer to this part on the top of the next page.)

Problem 2 (continued):

(b) Is it possible to change the constants on the right-hand side of the system so that the new system is inconsistent? In order to receive any credit, you must justify your answer.

3. (12 points) Let

$$\mathbf{M} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix},$$

and suppose that $\det \mathbf{M} = 4$. Calculate the determinants of the following matrices. You will not receive any credit unless you include a one-sentence justification of your answer.

(a) $\mathbf{A} = \begin{bmatrix} 3\mathrm{a} & 3\mathrm{b} & 3\mathrm{c} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \\ 2\mathrm{d} & 2\mathrm{e} & 2\mathrm{f} \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2\mathrm{a} & 2\mathrm{b} & 2\mathrm{c} \\ 2\mathrm{d} & 2\mathrm{e} & 2\mathrm{f} \\ 2\mathrm{g} & 2\mathrm{h} & 2\mathrm{i} \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 2\mathrm{d} + 3\mathrm{a} & 2\mathrm{e} + 3\mathrm{b} & 2\mathrm{f} + 3\mathrm{c} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{g} \\ \mathrm{b} + 3\mathrm{a} & \mathrm{e} + 3\mathrm{d} & \mathrm{h} + 3\mathrm{g} \\ \mathrm{c} & \mathrm{f} & \mathrm{i} \end{bmatrix}$

4. (18 points) **(Note that parts (d)–(f) of this question are on the next page.)** Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of V are subspaces of V? You will not receive any credit unless you justify your answer.

   (a) The set of all constant functions.

   (b) The set of all functions f such that $f(4) = 3$.

   (c) The set of all functions f such that $f(4) = 0$.

Problem 4 (continued):

(c) The set of all polynomials of degree 3.

(d) The set of all polynomials whose degree is at most 3.

(e) The set of all differentiable functions.

5. (12 points) For each of the following linear transformations T, determine if it is one-to-one and/or onto. In order to receive any credit, you must provide a brief justification for your answer.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$.

6. (30 points) Are the following statements true or false? You will not receive any credit unless you justify your answers. (Note that there are four more parts to this question on the next two pages.)

   (a) If there exists a linearly-dependent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V, then $\dim V \leq p - 1$.

   (b) If $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Question 6 (continued):

(c) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix $\mathbf{A}$ such that Col $\mathbf{A} = H$.

(d) If a matrix $\mathbf{U}$ has orthonormal columns, then $\mathbf{U}\mathbf{U}^T = \mathbf{I}$.

Question 6 (continued):

(e) If the square matrix $\mathbf{A}$ is row equivalent to the identity matrix $\mathbf{I}$, then $\mathbf{A}$ is diagonalizable.

(f) Let $\mathbf{A}$ be an $n \times n$ matrix. If the systems $\mathbf{A}\mathbf{x} = \mathbf{e}_j$ are consistent for $j = 1, 2, \ldots, n$, then $\mathbf{A}$ is invertible.