1. (12 points) Let

$$\mathbf{A} = \begin{bmatrix} -3 & 2 & -2 & 1 \\ 3 & 2 & -5 & 0 \\ -2 & -4 & 1 & -5 \\ 0 & 0 & 1 & -7 \end{bmatrix}.$$

Compute a basis for the $\lambda = -4$ eigenspace of $\mathbf{A}$.

$$A - (-4I) = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 0 \\ -2 & -4 & 5 & -5 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two free variables: x_2 and x_4

If $x_2 = 0$ and $x_4 = 1 \Rightarrow \begin{bmatrix} 5 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

If $x_2 = 1$ and $x_4 = 0 \Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\text{basis} = \left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

2. (16 points) Consider the system

$$\begin{aligned} x_1 + 3x_4 &= 4 \\ x_2 + x_3 - 3x_4 &= 2 \\ -x_1 - 3x_4 + x_5 &= -1 \\ 3x_1 + 9x_4 - 2x_5 &= 6 \end{aligned}$$

of four equations in five unknowns. 4x5 matrix A

(a) Express its solution set in parametric vector form.

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & -3 & 0 & 2 \\ -1 & 0 & 0 & -3 & 1 & -1 \\ 3 & 0 & 0 & 9 & -2 & 6 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -2 & -6 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 and x_4 are free variables

(There is extra space for your answer to this part on the top of the next page.)

Problem 2 (continued):

Soln set:
$$\begin{bmatrix} -3x_4+4 \\ -x_3+3x_4+2 \\ x_3 \\ x_4 \\ 3 \end{bmatrix} =$$

$$x_3\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

(b) Is it possible to change the constants on the right-hand side of the system so that the new system is inconsistent? In order to receive any credit, you must justify your answer.

Yes. We know that the column space of A is a 3-dim subspace of $\mathbb{R}^4$. If we let b be any vector in $\mathbb{R}^4$ that is not in the column space of A, then $Ax = b$ is inconsistent.

3. (12 points) Let

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

and suppose that $\det \mathbf{M} = 4$. Calculate the determinants of the following matrices. You will not receive any credit unless you include a one-sentence justification of your answer.

(a) $\mathbf{A} = \begin{bmatrix} 3a & 3b & 3c \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix}$

We obtain A from M by 3 row ops: $R_1 \to 3R_1$, $R_2 \to 2R_2$ and $R_2 \leftrightarrow R_3$

$\Rightarrow \det A = (-6)(4) = -24$

(b) $\mathbf{B} = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{bmatrix}$

We obtain B from M by three row ops: $R_1 \to 2R_1$, $R_2 \to 2R_2$, $R_3 \to 2R_3$

$\Rightarrow \det B = (2)^3(4) = 32$

(c) $\mathbf{C} = \begin{bmatrix} a & b & c \\ 2d+3a & 2e+3b & 2f+3c \\ g & h & i \end{bmatrix}$

We obtain C from M by two row ops: $R_2 \to 2R_2$, $R_2 \to R_2 + 3R_1$

$\Rightarrow \det C = (2)(4) = 8$

(d) $\mathbf{D} = \begin{bmatrix} a & d & g \\ b+3a & e+3d & h+3g \\ c & f & i \end{bmatrix}$

We obtain D from M by transposing and applying the row op $R_2 \to R_2 + 3R_1$

$\Rightarrow \det D = \det M = 4$

4. (18 points) **(Note that parts (d)–(f) of this question are on the next page.)** Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of V are subspaces of V? You will not receive any credit unless you justify your answer.

   (a) The set of all constant functions. Subspace

   If we add two functions that are constant, the result is a constant function. If we multiply a constant function by a (constant) scalar, we obtain a constant function.

   (b) The set of all functions f such that $f(4) = 3$. Not a subspace.

   Let f_1 and f_2 be two functions that take the value 3 at 4. Then $(f_1+f_2)(4) = 3+3 = 6$. The set is not closed under vector addition.

   (c) The set of all functions f such that $f(4) = 0$. Subspace.

   Let f_1 and f_2 be two functions such that $f(4)=0$. Then $(f_1+f_2)(4) = 0$. $\Rightarrow$ set is closed under vector addition. Let f be a function such that $f(4)=0$ and let r be a scalar. Then $(rf)(4) = r(f(4)) = (r)(0) = 0$. $\Rightarrow$ closure under scalar multiplication.

Problem 4 (continued):

(c) The set of all polynomials of degree 3. Not a subspace.

Consider the two polynomials

$p_1(t) = t^3 + 2t^2 + 4$ and $p_2(t) = -t^3 + t - 7$.

Then $(p_1 + p_2)(t) = 2t^2 + t - 3$. Since $p_1 + p_2$ is not a polynomial of degree 3, the set is not closed under vector addition

(d) The set of all polynomials whose degree is at most 3. Subspace

A polynomial of degree at most 3 can be written as $p(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$ where the coefficients can be any real numbers including 0. If two such polynomial are added together, we get another such polynomial. If such a polynomial is multiplied by a scalar we get another such polynomial.

(e) The set of all differentiable functions.

Subspace.

In calculus, we learn that the sum of two differentiable functions is differentiable and the product of a differentiable function and a constant is differentiable. This set is closed under vector addition and scalar multiplication.

5. (12 points) For each of the following linear transformations T, determine if it is one-to-one and/or onto. In order to receive any credit, you must provide a brief justification for your answer.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

T is one-to-one $\Leftrightarrow$ nul space of $T = \{0\}$.

$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 = 0 \Rightarrow x_1 = 0$

$3x_1 - 2x_3 = 0$ and $x_1 = 0 \Rightarrow x_3 = 0$

$x_1 - x_2 + x = 0$ and $x_1 = x_3 = 0 \Rightarrow x_2 = 0$

$\Rightarrow$ nul space of $T = \{0\} \Rightarrow T$ is one-to-one

rank T + dim nul $T = 3$ and dim nul $T = 0$

$\Rightarrow$ rank $T = 3 \Rightarrow T$ is onto.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$.

Image of T = Span$\{T(e_1), T(e_2), T(e_3)\}$.

Since $(1,3)$ and $(4,-7)$ are linearly independent in $\mathbb{R}^2$, they span $\mathbb{R}^2$.

$\Rightarrow$ Image of $T = \mathbb{R}^2$. $\Rightarrow T$ is onto.

(Rank T) + (dim nul T) $= 3$ and

Rank $T = 2 \Rightarrow$ dim nul $T = 1$.

$\Rightarrow T$ is not one-to-one.

6. (30 points) Are the following statements true or false? You will not receive any credit unless you justify your answers. (Note that there are four more parts to this question on the next two pages.)

   (a) If there exists a linearly-dependent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V, then $\dim V \leq p-1$.

False. The set $\{e_1, 2e_1\}$ is linearly dependent in $\mathbb{R}^2$, and $\dim \mathbb{R}^2 = 2$.

   (b) If $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

True. $\|u+v\|^2 = (u+v)\cdot(u+v)$

$= u \cdot u + 2(u\cdot v) + v \cdot v$

$\|u\|^2 = u \cdot u \qquad \|v\|^2 = v \cdot v$

So $\|u+v\|^2 = \|u\|^2 + \|v\|^2 \iff 2(u\cdot v) = 0$

$\iff u \cdot v = 0$

Question 6 (continued):

(c) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix $\mathbf{A}$ such that Col $\mathbf{A} = H$.

True. If $H = \{\text{zero vector}\}$, then $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Otherwise, $\dim H = 1, 2,$ or 3.

If $\dim H = 1$, let $\{b\}$ be a basis of H. Then $A = [\, b \,|\, b \,|\, b \,]$. If $\dim H = 2$, let $\{b_1, b_2\}$ be a basis, then $A = [\, b_1 \,|\, b_2 \,|\, b_2 \,]$.

If $\dim H = 3$, then $H = \mathbb{R}^3$ and $A = [\, e_1 \,|\, e_2 \,|\, e_3 \,] = I_3$

(d) If a matrix $\mathbf{U}$ has orthonormal columns, then $\mathbf{U}\mathbf{U}^T = \mathbf{I}$.

False. Let U be the 2×1 matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Then

$$UU^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(This is the projection matrix $P : \mathbb{R}^2 \to \mathbb{R}^2$ for projection onto the line $x_2 = x_1$.)

Question 6 (continued):

(e) If the square matrix $\mathbf{A}$ is row equivalent to the identity matrix $\mathbf{I}$, then $\mathbf{A}$ is diagonalizable.

False. A is row equivalent to I

$\iff$ A is invertible.

The matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is invertible

but not diagonalizable.

(The $\lambda = 2$ eigenspace is 1-dim.)

(f) Let $\mathbf{A}$ be an $n \times n$ matrix. If the systems $\mathbf{Ax} = \mathbf{e}_j$ are consistent for $j = 1, 2, \ldots, n$, then $\mathbf{A}$ is invertible.

True. Let v_j be a solution to the system $Ax = e_j$. Form the matrix $B = [v_1 | v_2 | \cdots | v_n]$.

Then $AB = [Av_1 | Av_2 | \cdots | Av_n]$

$= [e_1 | e_2 | \cdots | e_n] = I$.