Elementary matrices and computing inverses

Definition. An *elementary* matrix is a matrix that is obtained from the identity matrix by applying exactly one elementary row operation.

There are three types of elementary row operations—one for each type of row operation.

What happens to a matrix if we multiply it by an elementary matrix?

Example.

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algorithm for computing $\mathbf{A}^{-1}$

Form the augmented matrix $[\ \mathbf{A} \mid \mathbf{I}\ ]$. Row reduce this matrix so that the left half becomes the identity matrix. At that point, the right half is $\mathbf{A}^{-1}$.

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The Invertible Matrix Theorem

Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following twelve statements are equivalent:

(a) $\mathbf{A}$ is an invertible matrix.

(b) $\mathbf{A}$ is row equivalent to the identity matrix.

(c) $\mathbf{A}$ has n pivot positions

(d) The equation $\mathbf{Ax} = \mathbf{0}$ has no nontrivial solutions.

(e) The columns of $\mathbf{A}$ are linearly independent.

(f) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ is one-to-one.

(g) The equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.

(h) The columns of $\mathbf{A}$ span $\mathbb{R}^n$.

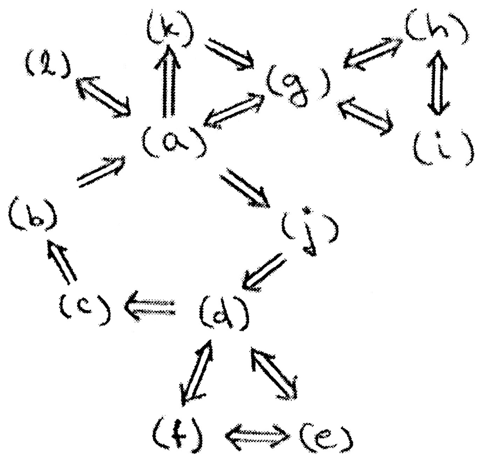
(i) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.

(k) There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}$.

(l) $\mathbf{A}^T$ is an invertible matrix.

Comments on the proof:



Determinants

We start with a recursive definition of the determinant.

Definition. The determinant of a 1×1 matrix $[a_{11}]$ is a_{11}.

Now we define the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices.

Definition. Given an $n \times n$ matrix $\mathbf{A}$, the ijth minor $\mathbf{A}_{ij}$ of $\mathbf{A}$ is the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by eliminating the ith row and jth column. The ijth cofactor of $\mathbf{A}$ is

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}.$$

Example. Compute the cofactors of the third column of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}.$$

Definition/Theorem. If $\mathbf{A}$ is an $n \times n$ matrix, the determinant of $\mathbf{A}$ can be computed using cofactor expansion along the ith row by

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

or by cofactor expansion along the jth column by

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Any row or any column yields the same result.

Example. Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}$$

by cofactor expansion along the third column.

Note that we get the familiar formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Is there a way to define $\det \mathbf{A}$ without recursion?