

## Determinants

We start with a recursive definition of the determinant.

**Definition.** The determinant of a  $1 \times 1$  matrix  $[a_{11}]$  is  $a_{11}$ .

Now we define the determinant of an  $n \times n$  matrix in terms of determinants of  $(n-1) \times (n-1)$  matrices.

**Definition.** Given an  $n \times n$  matrix  $\mathbf{A}$ , the  $ij$ th minor  $\mathbf{A}_{ij}$  of  $\mathbf{A}$  is the  $(n-1) \times (n-1)$  matrix obtained from  $\mathbf{A}$  by eliminating the  $i$ th row and  $j$ th column. The  $ij$ th cofactor of  $\mathbf{A}$  is

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}.$$

**Example.** Compute the cofactors of the third column of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}.$$

**Definition/Theorem.** If  $\mathbf{A}$  is an  $n \times n$  matrix, the determinant of  $\mathbf{A}$  can be computed using cofactor expansion along the  $i$ th row by

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

or by cofactor expansion along the  $j$ th column by

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}.$$

Any row or any column yields the same result.

**Example.** Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}$$

by cofactor expansion along the third column.

Note that we get the familiar formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Is there a way to define  $\det \mathbf{A}$  without recursion?

How do we go about computing  $\det \mathbf{A}$ ?

One type of matrix is perfectly suited for cofactor expansion.

**Theorem.** If  $\mathbf{A}$  is a triangular matrix, then  $\det \mathbf{A}$  is the product of its entries along the main diagonal.

## Properties of the Determinant

In order to gain some insight into how we will compute determinants in general, let's calculate the determinants of all elementary  $3 \times 3$  matrices.

**Theorem.** Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $n \times n$  matrices. Then

1. The matrix  $\mathbf{A}$  is invertible if and only if  $\det \mathbf{A} \neq 0$ .
2.  $\det \mathbf{A}^T = \det \mathbf{A}$
3.  $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$

Given the fact that  $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$ , we can consider the determinant of the product

$$\mathbf{EA}$$

where  $\mathbf{E}$  is an elementary matrix.

Row operations and the determinant:

1. Suppose that  $\mathbf{B}$  is obtained from  $\mathbf{A}$  by applying exactly one row replacement row operation, then

$$\det \mathbf{B} =$$

2. Suppose that  $\mathbf{B}$  is obtained from  $\mathbf{A}$  by applying exactly one row swap row operation, then

$$\det \mathbf{B} =$$

3. Suppose that  $\mathbf{B}$  is obtained from  $\mathbf{A}$  by applying exactly one row scaling row operation, then

$$\det \mathbf{B} =$$

**Corollary.** If  $\mathbf{A}$  has two identical rows, then

$$\det \mathbf{A} = 0.$$

Proof of the fact that doing a row replacement row operation does not change the determinant: Suppose that

$$\mathbf{B} = \left[ \begin{array}{c} R_1 \\ \hline \vdots \\ \hline R_i + \alpha R_j \\ \hline \vdots \\ \hline R_n \end{array} \right]$$

where  $R_1, R_2, \dots, R_n$  represent the rows of  $\mathbf{A}$ .

**Example.** Consider the  $4 \times 4$  matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 4 & 14 \\ 4 & 3 & 1 & 2 \\ -1 & 8 & 6 & 2 \\ 2 & -2 & 4 & -3 \end{bmatrix}$$

Let's calculate the determinant of  $\mathbf{A}$  using row operations.

Some practice with the properties of determinants:

Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $4 \times 4$  matrices with  $\det \mathbf{A} = 3$  and  $\det \mathbf{B} = -2$ . Compute:

1.  $\det \mathbf{AB}$
2.  $\det \mathbf{B}^5$
3.  $\det 2\mathbf{A}$
4.  $\det \mathbf{A}^T \mathbf{A}$
5.  $\det \mathbf{B}^{-1} \mathbf{AB}$