More on the definition of the determinant

Last class we started defining the determinant recursively.

Definition. The determinant of a 1×1 matrix $[a_{11}]$ is a_{11}.

Definition. Given an $n \times n$ matrix $\mathbf{A}$, the ijth minor $\mathbf{A}_{ij}$ of $\mathbf{A}$ is the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by eliminating the ith row and jth column. The ijth cofactor of $\mathbf{A}$ is

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}.$$

Example. We computed the cofactors of the third column of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}.$$

The $1,3$-minor is

$$\mathbf{A}_{1,3} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \end{bmatrix},$$

and the corresponding cofactor is $C_{1,3} = (+1) \det \mathbf{A}_{1,3} = 8$.

The $2,3$-minor is

$$\mathbf{A}_{2,3} = \begin{bmatrix} -1 & 4 \\ 4 & 0 \end{bmatrix},$$

and the corresponding cofactor is $C_{2,3} = (-1) \det \mathbf{A}_{2,3} = 16$.

The $3,3$-minor is

$$\mathbf{A}_{3,3} = \begin{bmatrix} -1 & 4 \\ 3 & -2 \end{bmatrix},$$

and the corresponding cofactor is $C_{3,3} = (+1) \det \mathbf{A}_{3,3} = -10$.

Definition/Theorem. If $\mathbf{A}$ is an $n \times n$ matrix, the determinant of $\mathbf{A}$ can be computed using cofactor expansion along the ith row by

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

or by cofactor expansion along the jth column by

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Cofactor expansion along any row or any column yields the same result.

Example. Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}$$

by cofactor expansion along the third column.

Note that we get the familiar formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Is there a way to define $\det \mathbf{A}$ without recursion?

How do we go about computing $\det \mathbf{A}$?

One type of matrix is perfectly suited for cofactor expansion.

Theorem. If $\mathbf{A}$ is a triangular matrix, then $\det \mathbf{A}$ is the product of its entries along the main diagonal.

Properties of the determinant

In order to gain some insight into how we will compute determinants in general, let's calculate the determinants of all elementary 3×3 matrices.

Theorem. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Then

1. The matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.
2. $\det \mathbf{A}^T = \det \mathbf{A}$
3. $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$

Given the fact that $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$, we can consider the determinant of the product

$$\mathbf{EA}$$

where $\mathbf{E}$ is an elementary matrix.

Row operations and the determinant:

1. Suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying exactly one row replacement row operation, then
$$\det \mathbf{B} =$$
2. Suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying exactly one row swap row operation, then
$$\det \mathbf{B} =$$
3. Suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by applying exactly one row scaling row operation, then
$$\det \mathbf{B} =$$

Corollary. If $\mathbf{A}$ has two identical rows, then $\det \mathbf{A} = 0$.

Proof of the fact that doing a row replacement row operation does not change the determinant: Suppose that

$$\mathbf{B} = \begin{bmatrix} R_1 \\ \hline \vdots \\ \hline R_i + \alpha R_j \\ \hline \vdots \\ \hline R_n \end{bmatrix}$$

where R_1, R_2, $\ldots$, R_n represent the rows of $\mathbf{A}$.

Example. Consider the 4×4 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 4 & 14 \\ 4 & 3 & 1 & 2 \\ -1 & 8 & 6 & 2 \\ 2 & -2 & 4 & -3 \end{bmatrix}$$

Let's calculate the determinant of $\mathbf{A}$ using row operations.

Some practice with the properties of determinants:

Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 matrices with $\det \mathbf{A} = 3$ and $\det \mathbf{B} = -2$. Compute:

1. $\det \mathbf{AB}$

2. $\det \mathbf{B}^5$

3. $\det 2\mathbf{A}$

4. $\det \mathbf{A}^{\mathrm{T}}\mathbf{A}$

5. $\det \mathbf{B}^{-1}\mathbf{AB}$