More on matrix multiplication

Last class we learned how to multiply two matrices. If $\mathbf{A}$ is an $m \times n$-matrix and $\mathbf{B}$ is an $n \times p$-matrix, then the product $\mathbf{C}$ of $\mathbf{A}$ and $\mathbf{B}$ is the matrix

$$\mathbf{C} = \left[\begin{array}{c|c|c|c} \mathbf{AB}_1 & \mathbf{AB}_2 & \ldots & \mathbf{AB}_p \end{array} \right]$$

where $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_p$ are the columns of $\mathbf{B}$. We also saw that the entries of $\mathbf{AB}$ satisfy the row-column rule

$$(\mathbf{AB})_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

As an example, we computed the product of $\mathbf{A}$ and $\mathbf{B}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

We obtained

$$\mathbf{AB} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \\ 76 & 100 \\ 103 & 136 \end{bmatrix}.$$

It is useful to remember that

$$\mathbf{AB} = \left[\begin{array}{c|c} \mathbf{A}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} & \mathbf{A}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \end{array} \right]$$

and that any particular entry in $\mathbf{AB}$ can be computed using the row-column rule. For example, the 3,2 entry of $\mathbf{AB}$ is

$$7 \cdot 2 + 8 \cdot 4 + 9 \cdot 6 = 100.$$

We should discuss some properties of matrix multiplication.

Definition. The $n \times n$ (square) matrix

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & \ldots & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & \ldots & 1 \end{bmatrix}$$

with 1's down the diagonal and 0's everywhere else is called the $n \times n$ *identity matrix.*

Theorem 2. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ and $\mathbf{C}$ be matrices of appropriate sizes. Then

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
3. $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
4. $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B} = \mathbf{A}(r\mathbf{B})$ for any scalar r
5. $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$

Three warnings.

1. $\mathbf{AB}$ does not always equal $\mathbf{BA}$. For example, consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}.$$

2. $\mathbf{AB} = \mathbf{AC}$ does not necessarily imply that $\mathbf{B} = \mathbf{C}$. For example, consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -1 & 4 \\ 4 & 3 \end{bmatrix}.$$

3. $\mathbf{AB} = \mathbf{0}$ does not necessarily imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. For example, consider

$$\mathbf{A} = \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}.$$

We will occasionally need to use the transpose of a matrix.

Definition. Given an $m \times n$ matrix $\mathbf{A}$, its transpose $\mathbf{A}^T$ is the $n \times m$ matrix such that

$$(\mathbf{A}^T)_{ij} = \mathbf{A}_{ji}.$$

Example. Consider $\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6\pi \end{bmatrix}$. Then $\mathbf{M}^T =$

Theorem 3. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices whose sizes are appropriate for the following sums and products. Then

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
3. $(r\mathbf{A})^T = r\mathbf{A}^T$
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Inverses of square matrices

Now we examine the question of whether or not a square matrix has a multiplicative inverse.

Definition. Suppose that $\mathbf{A}$ is an $n \times n$ matrix (a square matrix). If there exists an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I} \qquad \text{and} \qquad \mathbf{BA} = \mathbf{I},$$

then we say that $\mathbf{A}$ is *invertible* and that $\mathbf{B}$ is the *inverse* of $\mathbf{A}$.

Note: If such a matrix $\mathbf{B}$ exists, then it is unique.

Examples.

1. Consider $\mathbf{A}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

2. Consider $\mathbf{A}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

A typical square matrix $\mathbf{A}$ has a multiplicative inverse which we denote by $\mathbf{A}^{-1}$, so

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

However, there are infinitely many square matrices that do not have inverses.

For 2×2 matrices, there is a simple formula for $\mathbf{A}^{-1}$.

Theorem 4. Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $\mathbf{A}$ is not invertible.

Here are some basic properties of inverses.

Theorem 6.

1. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
2. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then $\mathbf{AB}$ is invertible. Moreover, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
3. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^T$ is invertible, and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Elementary matrices and computing inverses

Definition. An *elementary* matrix is a matrix that is obtained from the identity matrix by applying exactly one elementary row operation.

There are three types of elementary row operations—one for each type of row operation.

What happens to a matrix if we multiply it by an elementary matrix?

Example.

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$