One example of properties of the determinant

Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 matrices with $\det \mathbf{A} = 3$ and $\det \mathbf{B} = -2$. Compute:

1. $\det \mathbf{AB}$

2. $\det \mathbf{B}^5$

3. $\det 2\mathbf{A}$

4. $\det \mathbf{A}^{\mathrm{T}}\mathbf{A}$

5. $\det \mathbf{B}^{-1}\mathbf{AB}$

The classical adjoint and Cramer's Rule

Now we study two computationally oriented applications of the determinant which are sometimes theoretically useful.

Recall the definitions of cofactors and the determinant.

Definition. Given an $n \times n$ matrix $\mathbf{A}$, the ijth minor $\mathbf{A}_{ij}$ of $\mathbf{A}$ is the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by eliminating the ith row and jth column. The ijth cofactor of $\mathbf{A}$ is

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}.$$

Definition/Theorem. If $\mathbf{A}$ is an $n \times n$ matrix, the determinant of $\mathbf{A}$ can be computed using cofactor expansion along the ith row by

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

or by cofactor expansion along the jth column by

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Any row or any column yields the same result.

Both of these formulas are reminiscent of matrix multiplication except for the fact that the indexes are not quite right.

Given an $n \times n$ matrix $\mathbf{A}$, there is a cofactor C_{ij} for each entry of the matrix. Hence, we can make another $n \times n$ matrix $\mathbf{C}$ using the cofactors of $\mathbf{A}$.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Its matrix of cofactors is

$$\mathbf{C} = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let's multiply $\mathbf{A}$ and $\mathbf{C}^T$.

Why are the diagonal terms of $\mathbf{AC}^T$ the way that they are?

What about the off diagonal terms of $\mathbf{AC}^T$? For example, consider the $1,2$-term of the product $\mathbf{AC}^T$.

Definition. The *classical adjoint* $\operatorname{adj}\mathbf{A}$ of an $n \times n$ matrix $\mathbf{A}$ is the transpose of its matrix of cofactors. It is also called the adjugate of $\mathbf{A}$.

Theorem. For any $n \times n$ matrix $\mathbf{A}$, $\mathbf{A}(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})\,\mathbf{I}$. If $\det\mathbf{A} \neq 0$, then

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\operatorname{adj}\mathbf{A}.$$

Cramer's Rule

As we have seen, the linear system $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ if $\mathbf{A}$ is invertible. Using the formula above for $\mathbf{A}^{-1}$, we can derive an appealing formula for $\mathbf{A}^{-1}\mathbf{b}$.

Summary. (Cramer's Rule) The kth component x_k of the vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is given by

$$x_k = \frac{\det \mathbf{B}_k}{\det \mathbf{A}},$$

where the matrix $\mathbf{B}_k$ is the obtained from the matrix $\mathbf{A}$ by replacing the kth column of $\mathbf{A}$ with $\mathbf{b}$.

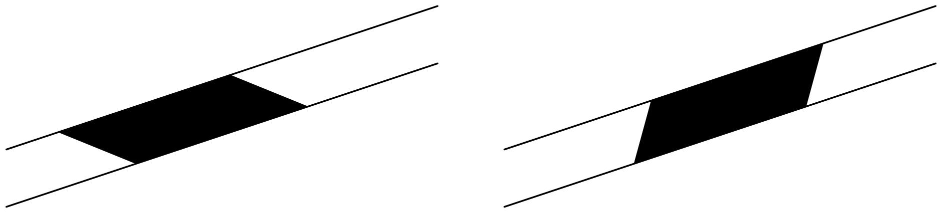
Example. Solve the system

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 4 \\ x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 - x_3 &= -1. \end{aligned}$$

Warning. Using the classical adjoint or Cramer's Rule usually involves a lot more work than the row reduction techniques that we have already discussed.

Area and the determinant

What can you say about the areas of the following two parallelograms?

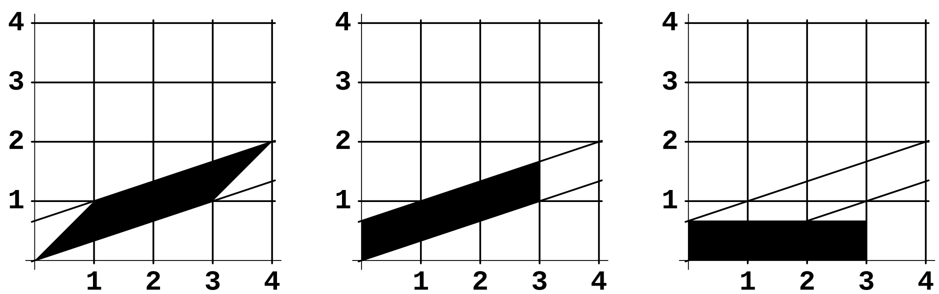


Consider a parallelogram $P = \{r_1\mathbf{u} + r_2\mathbf{v} \mid 0 \leq r_1 \leq 1,\ 0 \leq r_2 \leq 1\}$ in $\mathbb{R}^2$ formed by two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$.

Example. Let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let's calculate the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ in an unusual way.



We can represent these two geometric manipulations as column operations on a matrix.

Summary: 2×2 Matrices and Area in $\mathbb{R}^2$

Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

then the area of the parallelogram formed by **u** and **v** is

$$\left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|.$$

Volume in $\mathbb{R}^3$

The same basic geometric argument that works in $\mathbb{R}^2$ also works in $\mathbb{R}^3$. Given three linearly independent vectors **a**, **b**, and **c** in $\mathbb{R}^3$, consider the parallelopiped P that they determine.

How do column operations change the parallelopiped, and what do they do to the corresponding volumes?

Example. Consider the parallelopiped generated by

$$\mathbf{a} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix},$$

Summary: 3×3 Matrices and Volume in $\mathbb{R}^3$

Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then the volume of the parallelopiped formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is

$$\left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|.$$

Determinants and linear transformations

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a square matrix $\mathbf{A}$ such that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

What is the significance of $\det \mathbf{A}$ in this situation?

Consider the case where $n = 2$ and start with a parallelogram P determined by two vectors $\mathbf{u}$ and $\mathbf{v}$.

Summary: Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, then $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some 2×2 matrix $\mathbf{A}$. If P is a parallelogram determined by the two vectors $\mathbf{u}$ and $\mathbf{v}$, then $T(P)$ is also a parallelogram, and

$$\text{area of } T(P) = |\det A|(\text{area of } P).$$

For $n = 3$, the same conclusion holds if the concept of area is replaced by that of volume.

Also, there is nothing special about parallelograms in this discussion. We could just as well start with a region such as a disk.

For more details, see pp. 208–209 of our text.

If you have studied multivariable calculus, you know that there is a change of variables formula that is used to convert multiple integrals from one set of coordinates to another. That formula involves the determinant of the Jacobian matrix (see Stewart *Calculus: Concepts and Contexts*, Section 12.9). The area conversion formula mentioned here is a special case of that more general formula.