More on the geometry of the determinant

Volume in $\mathbb{R}^3$: The same basic geometric argument that works in $\mathbb{R}^2$ also works in $\mathbb{R}^3$. Given three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$, consider the parallelepiped

$$P = \{ r_1\mathbf{a} + r_2\mathbf{b} + r_3\mathbf{c} \mid 0 \le r_1 \le 1,\ 0 \le r_2 \le 1,\ 0 \le r_3 \le 1 \}$$

that they determine.

How do column operations change the parallelepiped, and what do they do to the corresponding volumes?

Example. Consider the parallelepiped generated by

$$\mathbf{a} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.$$

Summary: 3×3 Matrices and Volume in $\mathbb{R}^3$

Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is

$$\left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|.$$

Determinants and linear transformations

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a square matrix $\mathbf{A}$ such that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

What is the significance of $\det \mathbf{A}$ in this situation?

Consider the case where $n = 2$ and start with a parallelogram P determined by two vectors $\mathbf{u}$ and $\mathbf{v}$.

Example. Consider the parallelogram P determined by the two vectors

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

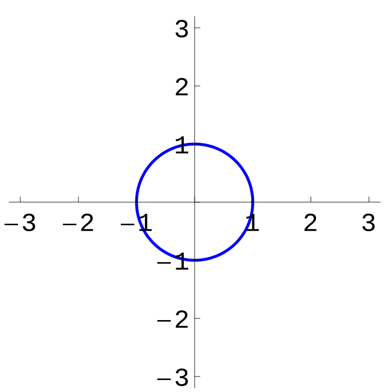
and the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$.

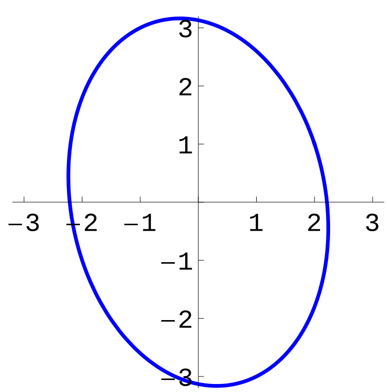
Summary: Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, then $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some 2×2 matrix $\mathbf{A}$. If P is a parallelogram determined by the two vectors $\mathbf{u}$ and $\mathbf{v}$, then $T(P)$ is also a parallelogram, and

$$\text{area of } T(P) = |\det A|(\text{area of } P).$$

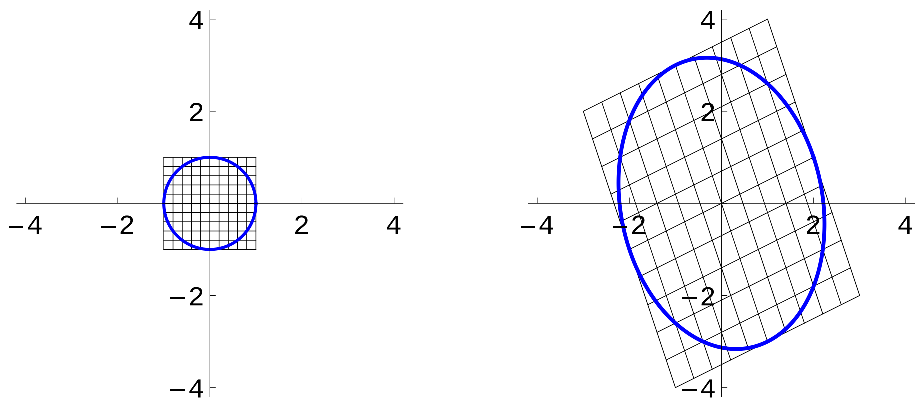
For $n = 3$, the same conclusion holds if the concept of area is replaced by that of volume.

Also, there is nothing special about parallelograms in this discussion. We could just as well start with a region such as a disk in $\mathbb{R}^2$. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ transforms the unit disk centered at the origin into an ellipse.





We can see how area is transformed by the linear transformation, we place a square grid over it, and we see how the grid is transformed.



If you have studied multivariable calculus, you know that there is a change of variables formula that is used to convert multiple integrals from one set of coordinates to another. That formula involves the determinant of the Jacobian matrix (see Stewart *Calculus: Concepts and Contexts*, Section 12.9 or Briggs and Cochran *Calculus*, Section 13.7). The area conversion formula mentioned here is a special case of that more general formula.

What is a vector space?

Definition. A vector space V is a set of objects that are called vectors along with two operations—vector addition and scalar multiplication. The vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is always defined for any pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, and given any scalar r in $\mathbb{R}$ and any vector $\mathbf{v}$ in V, the scalar multiple $r\mathbf{v}$ is a vector in V. Moreover, these two operations must satisfy the following eight properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For each $\mathbf{u}$, there is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Example 1. The vector space $\mathbb{R}^n$. See p. 27 of our text for a discussion of the properties listed above.

Example 2. The vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

Given two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, the vector sum $f + g$ of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

for all x in $\mathbb{R}$.

Given a function $f : \mathbb{R} \to \mathbb{R}$ and a real number r, the scalar multiple rf is defined by

$$(rf)(x) = r(f(x))$$

for all x in $\mathbb{R}$.

The operations of vector addition and scalar multiplication are illustrated by graphs that are posted on the course web site. In addition, at the same location, the concept of a linear combination of two functions is illustrated by two examples and their graphs.

Example 3. The vector space $M_{m\times n}$ of all $m \times n$ matrices. (We assume that the entries of the matrices are real numbers, but a different vector space is obtained if one allows the entries to be complex numbers.)

The operation of vector addition is the usual operation of addition for two matrices of the same size. The operation of scalar multiplication is the product of a real number and a matrix that was defined a couple of weeks ago.

The operations of vector addition and scalar multiplication and the concept of linear combination of two "vectors" are illustrated by examples that are posted on the course web site.

Example 4. The vector space $\mathbb{P}$ of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$.

Since polynomial functions qualify as functions as in Example 2, we can define vector addition and scalar multiplication just as we did in Example 2. Once again these operations are illustrated by formulas and graphs that are posted on the course web site.