More examples of vector spaces

Recall the definition of a vector space from last class:

Definition. A vector space V is a set of objects that are called vectors along with two operations—vector addition and scalar multiplication. The vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is always defined for any pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, and given any scalar r in $\mathbb{R}$ and any vector $\mathbf{v}$ in V, the scalar multiple $r\mathbf{v}$ is a vector in V. Moreover, these two operations must satisfy the following eight properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For each $\mathbf{u}$, there is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

We also discussed two examples of vector spaces last class—the usual $\mathbb{R}^n$ and the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

Here are two more examples:

Example 3. The vector space $M_{m\times n}$ of all $m \times n$ matrices. (We assume that the entries of the matrices are real numbers, but a different vector space is obtained if one allows the entries to be complex numbers.)

The operation of vector addition is the usual operation of addition for two matrices of the same size. The operation of scalar multiplication is the product of a real number and a matrix that was defined a couple of weeks ago.

The operations of vector addition and scalar multiplication and the concept of linear combination of two "vectors" are illustrated by examples that are posted on the course web site.

Example 4. The vector space $\mathbb{P}$ of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$.

Since polynomial functions qualify as functions as in Example 2, we can define vector addition and scalar multiplication just as we did in Example 2. Once again these operations are illustrated by formulas and graphs that are posted on the course web site.

Subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S,
2. (closure under vector addition) for each $\mathbf{v}_1$ and $\mathbf{v}_2$ in S, the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S, and
3. (closure under scalar multiplication) for each r in $\mathbb{R}$ and each $\mathbf{v}$ in S, the scalar multiple $r\mathbf{v}$ is in S.

Note. A subspace S of a vector space V is a vector space in its own right.

Example. Consider the line $x_2 = 3x_1$ in the vector space $\mathbb{R}^2$.

Example. Consider the line $x_2 = x_1 + 1$ in the vector space $\mathbb{R}^2$.

Example. Let $\mathbb{P}$ represent the vector space of all polynomial functions. Is $\mathbb{P}$ a subspace of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is S a subspace of $\mathbb{P}$?

Theorem. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ are vectors in a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Example. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of V are subspaces of V?

1. The set of all constant functions.

2. The set of all functions f such that $f(2) = 1$.

3. The set of all functions f such that $f(2) = 0$.

4. The set of all polynomials of degree 3.

5. The set of all polynomials whose degree is at most 3.

6. The set of all differentiable functions.

Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $\mathbf{A}$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ represent the columns of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \quad \ldots \quad & \mathbf{c}_n \end{array}\right].$$

These column vectors are vectors in $\mathbb{R}^m$.

Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ represent the rows of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{array}\right].$$

These row vectors are vectors in $\mathbb{R}^n$.

The column space of A. The column space of $\mathbf{A}$ is the span of the columns of $\mathbf{A}$. We write

$$\text{Col } \mathbf{A} = \text{Span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}.$$

The row space of A. The row space of $\mathbf{A}$ is the span of the rows of $\mathbf{A}$. We write

$$\text{Row } \mathbf{A} = \text{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}.$$

The null space of A. The null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{Ax} = \mathbf{0}.$$

The null space of $\mathbf{A}$ is denoted by Nul $\mathbf{A}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. The column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$, and the null space and the row space of $\mathbf{A}$ are subspaces of $\mathbb{R}^n$.

Application. Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.