Subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S,
2. (closure under vector addition) for each $\mathbf{v}_1$ and $\mathbf{v}_2$ in S, the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S, and
3. (closure under scalar multiplication) for each r in $\mathbb{R}$ and each $\mathbf{v}$ in S, the scalar multiple $r\mathbf{v}$ is in S.

Note. A subspace S of a vector space V is a vector space in its own right.

Example. Consider the line $x_2 = 3x_1$ in the vector space $\mathbb{R}^2$.

Example. Consider the line $x_2 = x_1 + 1$ in the vector space $\mathbb{R}^2$.

Example. Let $\mathbb{P}$ represent the vector space of all polynomial functions as discussed last class. Is $\mathbb{P}$ a subspace of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is S a subspace of $\mathbb{P}$?

Theorem. If $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_p$ are vectors in a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Example. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of V are subspaces of V?

1. The set of all constant functions.

2. The set of all functions f such that $f(2) = 1$.

3. The set of all functions f such that $f(2) = 0$.

4. The set of all polynomials of degree 3.

5. The set of all polynomials whose degree is at most 3.

6. The set of all differentiable functions.

Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $\mathbf{A}$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ represent the columns of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \quad \cdots \quad & \mathbf{c}_n \end{array} \right].$$

These column vectors are vectors in $\mathbb{R}^m$.

Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ represent the rows of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{array} \right].$$

These row vectors are vectors in $\mathbb{R}^n$.

The column space of A. The column space of $\mathbf{A}$ is the span of the columns of $\mathbf{A}$. We write

$$\text{Col } \mathbf{A} = \text{Span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}.$$

The row space of A. The row space of $\mathbf{A}$ is the span of the rows of $\mathbf{A}$. We write

$$\text{Row } \mathbf{A} = \text{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}.$$

The null space of A. The null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

The null space of $\mathbf{A}$ is denoted by Nul $\mathbf{A}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. The column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$, and the null space and the row space of $\mathbf{A}$ are subspaces of $\mathbb{R}^n$.

Application. Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 2 & -4 & 0 & 8 & 1 \end{bmatrix}.$$

Express the null space of $\mathbf{A}$ as the span of as few vectors as possible.

The consistency of a system of linear equations can be viewed as a statement about the column space of the coefficient matrix.

Fact. The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is an element of the column space of $\mathbf{A}$.

Here is how Lay (p. 232) contrasts Nul $\mathbf{A}$ and Col $\mathbf{A}$ for an $m \times n$ matrix $\mathbf{A}$:

Nul $\mathbf{A}$	Col $\mathbf{A}$
1. Nul $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.	1. Col $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.
2. Nul $\mathbf{A}$ is implicitly defined; that is, you are given only a condition ($\mathbf{Ax} = \mathbf{0}$) that vectors in Nul $\mathbf{A}$ must satisfy.	2. Col $\mathbf{A}$ is explicitly defined; that is, you are told how to build vectors in Col $\mathbf{A}$.
3. It takes time to find vectors in Nul $\mathbf{A}$. Row operations on $[\mathbf{A} \quad \mathbf{0}]$ are required.	3. It is easy to find vectors in Col $\mathbf{A}$. The columns of $\mathbf{A}$ are displayed; others are formed from them.
4. There is no obvious relation between Nul $\mathbf{A}$ and the entries in $\mathbf{A}$.	4. There is an obvious relation between Col $\mathbf{A}$ and the entries in $\mathbf{A}$, since each column of $\mathbf{A}$ is in Col $\mathbf{A}$.
5. A typical vector $\mathbf{v}$ in Nul $\mathbf{A}$ has the property that $\mathbf{Av} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col $\mathbf{A}$ has the property that the equation $\mathbf{Ax} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul $\mathbf{A}$. Just compute $\mathbf{Av}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col $\mathbf{A}$. Row operations on $[\mathbf{A} \quad \mathbf{v}]$ are required.
7. Nul $\mathbf{A} = \{\mathbf{0}\}$ if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.	7. Col $\mathbf{A} = \mathbb{R}^m$ if and only if the equation $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $\mathbf{A} = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one.	8. Col $\mathbf{A} = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.