Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $\mathbf{A}$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ represent the columns of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \quad \ldots \quad & \mathbf{c}_n \end{array} \right].$$

These column vectors are vectors in $\mathbb{R}^m$.

Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ represent the rows of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{array} \right].$$

These row vectors are vectors in $\mathbb{R}^n$.

The column space of A. The column space of $\mathbf{A}$ is the span of the columns of $\mathbf{A}$. We write

$$\text{Col } \mathbf{A} = \text{Span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}.$$

The row space of A. The row space of $\mathbf{A}$ is the span of the rows of $\mathbf{A}$. We write

$$\text{Row } \mathbf{A} = \text{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}.$$

The null space of A. The null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

The null space of $\mathbf{A}$ is denoted by Nul $\mathbf{A}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. The column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$, and the null space and the row space of $\mathbf{A}$ are subspaces of $\mathbb{R}^n$.

The consistency of a system of linear equations can be viewed as a statement about the column space of the coefficient matrix.

Fact. The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is an element of the column space of $\mathbf{A}$.

Here is how Lay (p. 204) contrasts Nul $\mathbf{A}$ and Col $\mathbf{A}$ for an $m \times n$ matrix $\mathbf{A}$:

Nul $\mathbf{A}$

1. Nul $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.
2. Nul $\mathbf{A}$ is implicitly defined; that is, you are given only a condition ($\mathbf{Ax} = \mathbf{0}$) that vectors in Nul $\mathbf{A}$ must satisfy.
3. It takes time to find vectors in Nul $\mathbf{A}$. Row operations on $[\mathbf{A} \quad \mathbf{0}]$ are required.
4. There is no obvious relation between Nul $\mathbf{A}$ and the entries in $\mathbf{A}$.
5. A typical vector $\mathbf{v}$ in Nul $\mathbf{A}$ has the property that $\mathbf{Av} = \mathbf{0}$.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul $\mathbf{A}$. Just compute $\mathbf{Av}$.
7. Nul $\mathbf{A} = \{\mathbf{0}\}$ if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
8. Nul $\mathbf{A} = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one.

Col $\mathbf{A}$

1. Col $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.
2. Col $\mathbf{A}$ is explicitly defined; that is, you are told how to build vectors in Col $\mathbf{A}$.
3. It is easy to find vectors in Col $\mathbf{A}$. The columns of $\mathbf{A}$ are displayed; others are formed from them.
4. There is an obvious relation between Col $\mathbf{A}$ and the entries in $\mathbf{A}$, since each column of $\mathbf{A}$ is in Col $\mathbf{A}$.
5. A typical vector $\mathbf{v}$ in Col $\mathbf{A}$ has the property that the equation $\mathbf{Ax} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col $\mathbf{A}$. Row operations on $[\mathbf{A} \quad \mathbf{v}]$ are required.
7. Col $\mathbf{A} = \mathbb{R}^m$ if and only if the equation $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Col $\mathbf{A} = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Item 8 in both lists suggest two subspaces that are intimately connected with any linear transformation from one vector space to another.

Definition. A transformation $L : V_1 \to V_2$ from a vector space V_1 to a vector space V_2 is linear if

1. $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ for all vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V_1, and
2. $L(r\mathbf{v}) = rL(\mathbf{v})$ for all $\mathbf{v}$ in V_1 and all r in $\mathbb{R}$.

Example. Let V_1 be the vector space of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and let V_2 be the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. The operation of differentiation is a linear transformation from V_1 to V_2. That is, the transformation $D : V_1 \to V_2$ given by

$$D(f) = f'$$

is a linear transformation.

Associated to any linear transformation are two important subspaces.

Definition. The kernel of $L : V_1 \to V_2$ is the subset of $\mathbf{V}_1$ given by

$$\{\mathbf{v}_1 \,|\, L(\mathbf{v}_1) = \mathbf{0}\}.$$

The range of L is the subset of $\mathbf{V}_2$ given by

$$\{\mathbf{v}_2 \,|\, L(\mathbf{v}_1) = \mathbf{v}_2 \text{ for some } \mathbf{v}_1 \text{ in } V_1\}.$$

Fact. Both the kernel and the range of a linear transformation are subspaces. The kernel is a subspace of V_1, and the range is a subspace of V_2.

For a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ determined by the matrix $\mathbf{A}$, its range is Col $\mathbf{A}$, and its kernel is Nul $\mathbf{A}$.

Example. What are the kernel and range of the transformation $p : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}?$$

(This example was first introduced on September 20.)

Example. What are the kernel and the range of the differentiation transformation D mentioned above?

If we are careful, we can also use integration to define a linear transformation.

Example. Given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ in $\mathbb{P}_n$ (the vector space of all polynomial functions of degree at most n), we can define

$$I(p) = \int_0^x p(t)\,dt = a_n \frac{x^{n+1}}{n+1} + a_{n-1}\frac{x^n}{n} + \ldots + a_0 x.$$

The map $I : \mathbb{P}_n \to \mathbb{P}_{n+1}$ is a linear transformation. What are its kernel and range?