Bases for vector spaces and subspaces

Given a vector space or subspace $V$, we often find it convenient to express it as the span of a few vectors. A basis for $V$ is a spanning set that contains as few vectors as possible.

**Definition.** A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is a *basis* for $V$ if

1. it is linearly independent, and
2. it spans $V$.

**Example.** The standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$.

**Example.** The two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^2$. 
Example. The set \( \{x^3, x^2, x, 1\} \) is a basis of \( \mathbb{P}_3 \).
Example. The set \( \{x^3, x^3 + x^2, x, 1\} \) is another basis of \( \mathbb{P}_3 \).

We need ways of determining bases of vector spaces and their subspaces. The “casting-out procedure” produces a basis from a spanning set.

The casting-out procedure

Given a vector subspace \( S \) spanned by \( \{v_1, v_2, \ldots, v_k\} \), we can obtain a basis \( B \) for \( S \) by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. \( B_1 = \{v_1\} \) as long as \( v_1 \neq \mathbf{0} \), and
2. for \( i \geq 2 \),

(a) (cast out) \( B_i = B_{i-1} \) if \( v_i \) is in \( \text{Span } B_{i-1} \), or
(b) (keep) \( B_i = B_{i-1} \cup \{v_i\} \) if \( v_i \) is not in \( \text{Span } B_{i-1} \).

Then the final result \( B_k \) is a basis \( B \) for \( S \).
Example. Let’s apply the casting-out procedure to the set \( \{x^3 + 1, x, x^2, x^2 - x, 4, x^3\} \) of polynomials in \( \mathbb{P}_3 \).

Theorem (similar to The Spanning Set Theorem, Lay, p. 210). Let \( S = \text{Span}\{v_1, \ldots, v_k\} \). Then the final result \( B_k \) of the casting-out procedure applied to \( \{v_1, \ldots, v_k\} \) is a basis for \( S \).

The proof of the casting-out procedure is posted on the web site, and we will not discuss it in class.
A basis for the column space of a matrix

**Example.** Find a basis for the column space of

\[
A = \begin{bmatrix}
1 & -2 & 0 & 1 \\
0 & 0 & 3 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

**Fact:** Suppose that \( A \) and \( B \) are row equivalent matrices. Then the linear dependence relations among the columns of \( A \) are the same as the linear dependence relations among the columns of \( B \).

Why?
Example. Find a basis for the column space of

\[
B = \begin{bmatrix}
1 & -2 & 0 & 1 \\
-1 & 2 & 3 & 1 \\
0 & 0 & -3 & -2
\end{bmatrix}.
\]

Warning: If you row reduce a matrix \(B\) to a matrix \(A\) in row echelon form, you identify the pivot columns of \(B\). To find a basis for \(\text{Col } B\), use the pivot columns of \(B\). Do not use the pivot columns of \(A\). Row reduction usually changes the column space of a matrix.