Bases for vector spaces and subspaces

Given a vector space or subspace $V$, we often find it convenient to express it as the span of a few vectors. A basis for $V$ is a spanning set that contains as few vectors as possible.

**Definition.** A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a *basis* for $V$ if

1. it is linearly independent, and
2. it spans $V$.

**Example.** The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

**Example.** The two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^2$.

**Example.** The set $\{x^3, x^2, x, 1\}$ is a basis of $\mathbb{P}_3$.

**Example.** The set $\{x^3, x^3 + x^2, x, 1\}$ is another basis of $\mathbb{P}_3$.

We need ways of determining bases of vector spaces and their subspaces. The "casting-out procedure" produces a basis from a spanning set.

The casting-out procedure

Given a vector subspace $S$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can obtain a basis $B$ for $S$ by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
   (a) (cast out) $B_i = B_{i-1}$ if $\mathbf{v}_i$ is in $\operatorname{Span} B_{i-1}$, or
   (b) (keep) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if $\mathbf{v}_i$ is not in $\operatorname{Span} B_{i-1}$.

Then the final result $B_k$ is a basis $B$ for $S$.

**Example.** Let's apply the casting-out procedure to the set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ of polynomials in $\mathbb{P}_3$.

**Theorem** (similar to The Spanning Set Theorem, Lay, p. 210). Let $S = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then the final result $B_k$ of the casting-out procedure applied to $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $S$.

The proof of the casting-out procedure is posted on the web site, and we will not discuss it in class.

A basis for the column space of a matrix

**Example.** Find a basis for the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Fact:** Suppose that $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices. Then the linear dependence relations among the columns of $\mathbf{A}$ are the same as the linear dependence relations among the columns of $\mathbf{B}$.

Why?

**Example.** Find a basis for the column space of

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$

**Warning:** If you row reduce a matrix **B** to a matrix **A** in row echelon form, you identify the pivot columns of **B**. To find a basis for Col **B**, use the pivot columns of **B**. **Do not use the pivot columns of A.** Row reduction usually changes the column space of a matrix.