Bases for vector spaces and subspaces

A basis for V is a spanning set that contains as few vectors as possible.

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a *basis* for V if

1. it is linearly independent, and
2. it spans V.

Example. The set $\{x^3, x^2, x, 1\}$ is a basis of $\mathbb{P}_3$.

Example. The set $\{x^3, x^3 + x^2, x, 1\}$ is another basis of $\mathbb{P}_3$.

We need ways of determining bases of vector spaces and their subspaces. The "casting-out procedure" produces a basis from a spanning set.

The casting-out procedure

Given a vector subspace S spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can obtain a basis B for S by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
   (a) (cast out) $B_i = B_{i-1}$ if $\mathbf{v}_i$ is in $\operatorname{Span} B_{i-1}$, or
   (b) (keep) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if $\mathbf{v}_i$ is not in $\operatorname{Span} B_{i-1}$.

Then the final result B_k is a basis B for S.

Example. Let's apply the casting-out procedure to the set $\{x^3+1, x, x^2, x^2-x, 4, x^3\}$ of polynomials in $\mathbb{P}_3$.

Theorem. (similar to The Spanning Set Theorem, Lay, p. 239) Let $S = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Then the final result B_k of the casting-out procedure applied to $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for S.

The proof of the casting-out procedure is posted on the web site, and we will not discuss it in class. However, to understand the proof of the theorem, it is helpful to consider the example above along with the sets B_1, B_2, $\dots$, B_6. We get

$$B_1 = \left\{x^3+1\right\}$$

$$B_2 = \left\{x^3+1, x\right\}$$

$$B_3 = \left\{x^3+1, x, x^2\right\}$$

$$B_4 = B_3$$

$$B_5 = \left\{x^3+1, x, x^2, 4\right\}$$

$$B_6 = B_5.$$

A basis for Col $\mathbf{A}$

Example. Find a basis for the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example. Find a basis for the column space of

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$

Fact: Suppose that **A** and **B** are row equivalent matrices. Then the linear dependence relations among the columns of **A** are the same as the linear dependence relations among the columns of **B**.

Why?

Warning: If you row reduce a matrix **B** to a matrix **A** in row echelon form, you identify the pivot columns of **B**. To find a basis for Col **B**, use the pivot columns of **B**. **Do not use the pivot columns of A.** Row reduction usually changes the column space of a matrix.

Coordinates relative to a basis

A basis for a vector space produces a coordinate system for that space.

Theorem. (Unique Representation Theorem) Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then every vector $\mathbf{v}$ in V can be represented uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

The scalars $c_1, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to the basis B.

Example. Consider the spanning set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ for the vector space $\mathbb{P}_3$. There are infinitely many ways to write a given element of $\mathbb{P}_3$ as a linear combination of these vectors. For example, consider the polynomial $2x^3 - x^2$. It can be written as

$$(-1)x^2 + 2x^3.$$

It can also be written as $2(x^3+1)+(-1)x+(-1)(x^2-x)+(-\frac{1}{2})(4)$. Because this spanning set is not linearly independent, there are many ways to represent $2x^3-x^2$ as a linear combination of the vectors.

We produced a basis of $\mathbb{P}_3$ from this spanning set using the casting-out procedure. The basis is $\{x^3 + 1, x, x^2, 4\}$. What are the coordinates of $2x^3 - x^2$ relative to this basis?

Why are coordinates relative to a given basis unique?

The same vector has different coordinates relative to different bases.

Example. Consider the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$. What are its coordinates relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and what are its coordinates relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}?$$

Notation. Given the representation $\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ relative to the basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then the coordinates can be viewed as a vector in $\mathbb{R}^n$. This vector is denoted

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For example,

$$\left[\begin{bmatrix} -1 \\ -3 \end{bmatrix} \right]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

for the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Example. What is the coordinate vector for the cubic polynomial $2x^3 - x^2$ relative to the basis $B_1 = \{1, x, x^2, x^3\}$ of $\mathbb{P}_3$? What is its coordinate vector relative to the basis $B_2 = \{x^3 + 1, x, x^2, 4\}$?

The change of coordinates matrix for a basis B of $\mathbb{R}^n$

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$, then the B-coordinates of a vector $\mathbf{x}$ are related to the standard coordinates of $\mathbf{x}$ by the equation

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\mathbf{x} = \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{P}_B \left[\mathbf{x}\right]_B$$

where $\mathbf{P}_B$ is the matrix

$$\mathbf{P}_B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{array} \right].$$

Since $\mathbf{P}_B$ is invertible, we also have $\left[\mathbf{x}\right]_B = \left(\mathbf{P}_B\right)^{-1} \mathbf{x}$.

Example. We can double check our computation of the B-coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$ relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

using these equations.