

Coordinates relative to a basis

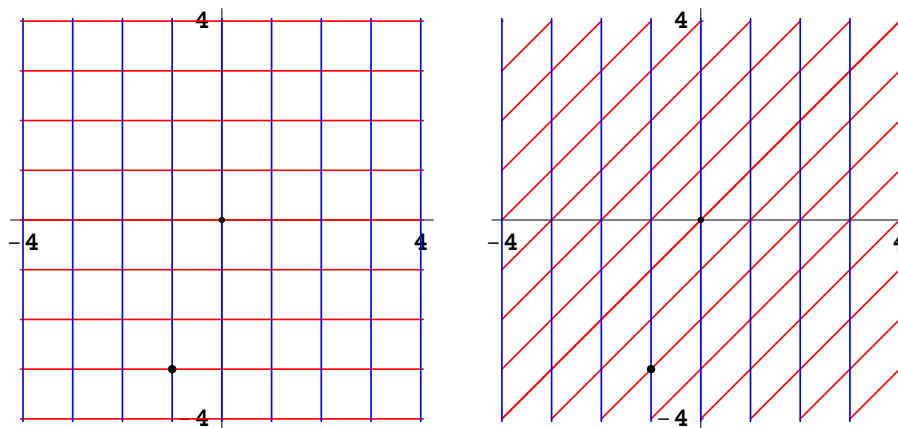
A basis for a vector space produces a coordinate system for that space. For example, consider the two bases

$$\{\mathbf{e}_1, \mathbf{e}_2\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

of \mathbb{R}^2 and the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} = (-1)\mathbf{e}_1 + (-3)\mathbf{e}_2 = (2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The weights in each linear combination are called the coordinates of \mathbf{x} relative to the given basis.



Theorem. (Unique Representation Theorem) Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then every vector \mathbf{v} in V can be represented uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n.$$

The scalars c_1, \dots, c_n are called the coordinates of \mathbf{v} relative to the basis B .

Example. Consider the basis $\{x^3 + 1, x, x^2, 4\}$ of \mathbb{P}_3 that we discussed last class. What are the coordinates of $2x^3 - x^2$ relative to this basis?

Example. Consider the spanning set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ for the vector space \mathbb{P}_3 . There are infinitely many ways to write a given element of \mathbb{P}_3 as a linear combination of these vectors. For example, consider the polynomial $2x^3 - x^2$. It can be written as

$$(-1)x^2 + 2x^3.$$

It can also be written as $2(x^3 + 1) + (-1)x + (-1)(x^2 - x) + (-\frac{1}{2})(4)$. Because this spanning set is not linearly independent, there are many ways to represent $2x^3 - x^2$ as a linear combination of the vectors.

Why are coordinates relative to a given basis unique?

Notation. Given the representation $\mathbf{v} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$ relative to the basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then the coordinates can be viewed as a vector in \mathbb{R}^n . This vector is denoted

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For example,

$$\left[\begin{bmatrix} -1 \\ -3 \end{bmatrix} \right]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

for the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Example. What is the coordinate vector for the cubic polynomial $2x^3 - x^2$ relative to the basis $B_1 = \{1, x, x^2, x^3\}$ of \mathbb{P}_3 ? What is its coordinate vector relative to the basis $B_2 = \{x^3 + 1, x, x^2, 4\}$?

The change of coordinates matrix for a basis B of \mathbb{R}^n

If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of \mathbb{R}^n , then the B -coordinates of a vector \mathbf{x} are related to the standard coordinates of \mathbf{x} by the equation

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\mathbf{x} = \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{P}_B [\mathbf{x}]_B$$

where \mathbf{P}_B is the matrix

$$\mathbf{P}_B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{array} \right].$$

Since \mathbf{P}_B is invertible, we also have $[\mathbf{x}]_B = (\mathbf{P}_B)^{-1} \mathbf{x}$.

Example. We can double check our computation of the B -coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in \mathbb{R}^2 relative to the basis $B = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$ using these matrices.

Coordinate transformations $V \rightarrow \mathbb{R}^n$

For any vector space V with basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, the B -coordinates define a nice linear transformation from V onto \mathbb{R}^n . It is defined by

$$\mathbf{v} \mapsto [\mathbf{v}]_B.$$

Theorem. The coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_B$ is a one-to-one linear transformation that maps V onto \mathbb{R}^n .

Definition. A one-to-one linear transformation that maps a vector space V onto a vector space W is called an isomorphism between V and W .

From the vector space point of view, two isomorphic vector spaces have the same structure.

Example. For what n is \mathbb{R}^n isomorphic to \mathbb{P}_3 ?

Example. For what n is \mathbb{R}^n isomorphic to the vector space $M_{2 \times 3}$ of 2×3 matrices?

The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

Definition. A vector space V is finite dimensional if it contains a finite spanning set. Otherwise, V is said to be infinite dimensional.

Example. The vector space \mathbb{R}^n is spanned by the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Therefore, it is finite dimensional.

Example. The vector space \mathbb{P}_3 of all polynomial functions whose degree is at most three is spanned by the basis $\{1, x, x^2, x^3\}$. Therefore, it is finite dimensional.

Example. \mathbb{P} is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)

Theorem. Let V be a vector space. Any finite spanning set for V has at least as many elements as any linearly independent subset of V .

Corollary. Any two bases of a finite-dimensional vector space V have the same number of elements.

Definition. The dimension of a finite-dimensional vector space V is the number of elements in any basis of V . This nonnegative integer is denoted $\dim V$.

Examples.

1. $\dim \mathbb{R}^n = n$
2. Let P be the plane $x_1 + x_2 + x_3 = 0$ in \mathbb{R}^3 . A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Hence, $\dim P = 2$.

3. $\dim \mathbb{P}_3 = 4$
4. $\dim M_{2 \times 3} = 6$

Here are a couple of other consequences of the notion of dimension.

Theorem. If $\dim V = n$, then any set in V with more than n vectors must be linearly dependent.

Theorem. If H is a subspace of V , then $\dim H \leq \dim V$. In fact, any basis of H can be expanded to a basis of V .

Suppose that \mathbf{A} is an $m \times n$ matrix. How can we determine the dimensions of $\text{Col } \mathbf{A}$ and $\text{Nul } \mathbf{A}$?

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}.$$

What relationship is there between the dimensions of $\text{Col } \mathbf{A}$ and $\text{Nul } \mathbf{A}$?