Coordinates relative to a basis

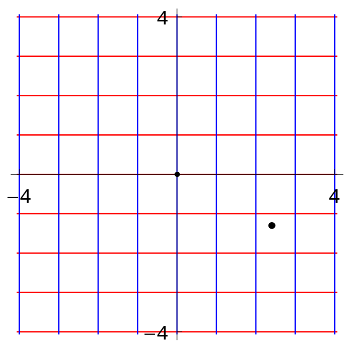
A basis for a vector space produces a coordinate system for that space. For example, consider the two bases

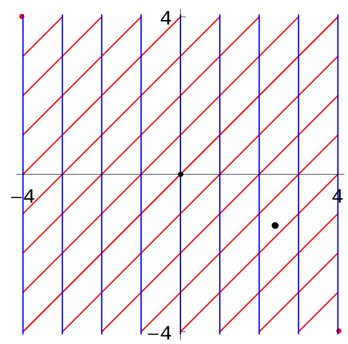
$$\{\mathbf{e}_1, \mathbf{e}_2\} \qquad \text{and} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^2$ and the vector

$$\mathbf{x} = \begin{bmatrix} 2.4 \\ -1.3 \end{bmatrix} = (2.4)\mathbf{e}_1 + (-1.3)\mathbf{e}_2 = (3.7) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1.3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The weights in each linear combination are called the coordinates of $\mathbf{x}$ relative to the given basis.





Theorem. (Unique Representation Theorem) Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then every vector $\mathbf{v}$ in V can be represented uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

The scalars $c_1, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to the basis B.

Example. Consider the basis $\{x^3 + 1, x, x^2, 4\}$ of $\mathbb{P}_3$ that we discussed last class. What are the coordinates of $2x^3 - x^2$ relative to this basis?

Example. Consider the spanning set $\{x^3+1, x, x^2, x^2-x, 4, x^3\}$ for the vector space $\mathbb{P}_3$. There are infinitely many ways to write a given element of $\mathbb{P}_3$ as a linear combination of these vectors. For example, consider the polynomial $2x^3 - x^2$. It can be written as

$$(-1)x^2 + 2x^3.$$

It can also be written as $2(x^3+1)+(-1)x+(-1)(x^2-x)+(-\frac{1}{2})(4)$. Because this spanning set is not linearly independent, there are many ways to represent $2x^3-x^2$ as a linear combination of the vectors.

Why are the coordinates relative to a given basis unique?

Notation. Given the representation $\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ relative to the basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then the coordinates can be viewed as a vector in $\mathbb{R}^n$. This vector is denoted

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For example,

$$\left[\begin{bmatrix} 2.4 \\ -1.3 \end{bmatrix}\right]_B = \begin{bmatrix} 3.7 \\ -1.3 \end{bmatrix}$$

for the basis

$$B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

Example. What is the coordinate vector for the cubic polynomial $2x^3 - x^2$ relative to the basis $B_1 = \{1, x, x^2, x^3\}$ of $\mathbb{P}_3$? What is its coordinate vector relative to the basis $B_2 = \{x^3 + 1, x, x^2, 4\}$?

The change of coordinates matrix for a basis B of $\mathbb{R}^n$

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$, then the B-coordinates of a vector $\mathbf{x}$ are related to the standard coordinates of $\mathbf{x}$ by the equation

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\mathbf{x} = \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{P}_B \, [\mathbf{x}]_B$$

where $\mathbf{P}_B$ is the matrix

$$\mathbf{P}_B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{array} \right].$$

Since $\mathbf{P}_B$ is invertible, we also have $[\mathbf{x}]_B = (\mathbf{P}_B)^{-1}\,\mathbf{x}$.

Example. We can double check our computation of the B-coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} 2.4 \\ -1.3 \end{bmatrix}$$

in $\mathbb{R}^2$ relative to the basis $B = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$ using these matrices.

Coordinate transformations $V \to \mathbb{R}^n$

For any vector space V with basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, the B-coordinates define a nice linear transformation from V onto $\mathbb{R}^n$. It is defined by

$$\mathbf{v} \mapsto [\mathbf{v}]_B.$$

Theorem. The coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_B$ is a one-to-one linear transformation that maps V onto $\mathbb{R}^n$.

Definition. A one-to-one linear transformation that maps a vector space V onto a vector space W is called an isomorphism between V and W.

From the vector space point of view, two isomorphic vector spaces have the same structure.

Example. For what n is $\mathbb{R}^n$ isomorphic to $\mathbb{P}_3$?

Example. For what n is $\mathbb{R}^n$ isomorphic to the vector space $M_{2\times 3}$ of 2×3 matrices?