The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

Definition. A vector space V is finite dimensional if it contains a finite spanning set. Otherwise, V is said to be infinite dimensional.

Example. The vector space $\mathbb{R}^n$ is spanned by the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Therefore, it is finite dimensional.

Example. The vector space $\mathbb{P}_3$ of all polynomial functions whose degree is at most three is spanned by the basis $\{1, x, x^2, x^3\}$. Therefore, it is finite dimensional.

Example. $\mathbb{P}$ is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)

Theorem. Let V be a vector space. Any finite spanning set for V has at least as many elements as any linearly independent subset of V.

Corollary. Any two bases of a finite-dimensional vector space V have the same number of elements.

Definition. The dimension of a finite-dimensional vector space V is the number of elements in any basis of V. This nonnegative integer is denoted $\dim V$.

Examples.

1. $\dim \mathbb{R}^n = n$
2. Let P be the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

   Hence, $\dim P = 2$.

3. $\dim \mathbb{P}_3 = 4$
4. $\dim M_{2\times 3} = 6$

Here are a couple of other consequences of the notion of dimension.

Theorem. If $\dim V = n$, then any set in V with more than n vectors must be linearly dependent.

Theorem. If H is a subspace of V, then $\dim H \leq \dim V$. In fact, any basis of H can be expanded to a basis of V.

Suppose that $\mathbf{A}$ is an $m \times n$ matrix. How can we determine the dimensions of Col $\mathbf{A}$ and Nul $\mathbf{A}$?

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}.$$

What relationship is there between the dimensions of Col $\mathbf{A}$ and Nul $\mathbf{A}$?

The rank of a matrix

Recall that the row space of an $m \times n$ matrix $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$.

Theorem. If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then Row $\mathbf{A}$ = Row $\mathbf{B}$.

How do we find a basis for Row $\mathbf{A}$?

Example. We know that the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent.

For an $m \times n$ matrix $\mathbf{A}$, we saw that

$$\dim(\text{Col } \mathbf{A}) + \dim(\text{Nul } \mathbf{A}) = n.$$

How is the dimension of Row $\mathbf{A}$ related to these numbers?

Definition. The rank of an $m \times n$ matrix $\mathbf{A}$ is the dimension of its column space. This dimension also equals the dimension of Row $\mathbf{A}$.

Example. Suppose that a homogeneous linear system of 10 equations in 6 unknowns has two linearly independent solutions and all other solutions are linear combinations of these. Can the solution set be described with fewer equations? If so, how many?

More equivalent conditions can be added to the Invertible Matrix Theorem.

Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following seven statements are equivalent.

1. The matrix $\mathbf{A}$ is invertible.
2. The columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$.
3. $\text{Col } \mathbf{A} = \mathbb{R}^n$
4. $\dim(\text{Col } \mathbf{A}) = n$
5. $\text{rank}(\mathbf{A}) = n$
6. $\text{Nul } \mathbf{A} = \{\mathbf{0}\}$
7. $\dim(\text{Nul } \mathbf{A}) = 0$

Since $\mathbf{A}$ is invertible if and only if $\mathbf{A}^T$ is invertible, any statement regarding the columns of $\mathbf{A}$ in the Invertible Matrix Theorem can be replaced by a statement regarding the rows of $\mathbf{A}$.

Rank and linear transformations

Consider a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{A}$ be its standard matrix representation. How is the rank of $\mathbf{A}$ related to the mapping properties of L?

Note: The range of L is the column space of $\mathbf{A}$ in $\mathbb{R}^m$. Consequently, $\text{rank}(\mathbf{A})$ is the dimension of the range of L. This number is often called the rank of L.

Example. Consider a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$. What is its rank?

Such a transformation is said to be of "full rank."

Example. What is the rank of the linear transformation $P : \mathbb{R}^2 \to \mathbb{R}^2$ that projects $\mathbb{R}^2$ onto the line $x_2 = -x_1$?

Example. Consider the linear transformation $p : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix

$$\mathbf{P} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(This example was first introduced on September 20 and discussed again on October 30.) What is its rank?

Theorem. Let $\mathbf{A}$ and $\mathbf{B}$ be transformations/matrices such that $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$. Then

$$\operatorname{rank}(\mathbf{AB}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$