More on eigenvectors and eigenvalues

Last class we discussed eigenvectors and eigenvalues for square matrices. Recall that the nonzero vector $\mathbf{x}$ is an eigenvector for the matrix $\mathbf{A}$ associated to the eigenvalue λ if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

We observed two basic facts about eigenvalues and eigenvectors.

1. The eigenvalues of the matrix $\mathbf{A}$ are the roots of its characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$. In other words, λ is an eigenvalue of $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

2. If λ is an eigenvalue for the matrix $\mathbf{A}$, then its eigenspace is the null space Nul $(\mathbf{A} - \lambda\mathbf{I})$.

We also calculated the characteristic polynomial of two 2×2 examples by hand.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 3\lambda - 2$. This quadratic does not factor easily, so we determined the eigenvalues using the quadratic formula. They are

$$\lambda = \frac{3 \pm \sqrt{17}}{2},$$

which are approximately 3.56 and -0.56 to two decimal places.

For larger matrices, calculating the characteristic polynomial can be time consuming. So I will use the computer to examine the characteristic polynomial for two larger matrices.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Last class we saw that the eigenvalues of $\mathbf{A}$ are $\lambda = 1$ and $\lambda = 2$. Moreover, we calculated that

$$\dim(\text{Nul }(\mathbf{A} - \mathbf{I})) = 2 \quad \text{and} \quad \dim(\text{Nul }(\mathbf{A} - 2\mathbf{I})) = 1.$$

The characteristic polynomial for this example is $-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (-1)(\lambda - 1)^2(\lambda - 2)$.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & -2 \\ -2 & 2 & -2 & 0 \\ -2 & 5 & 4 & -4 \\ 3 & 6 & -6 & -6 \end{bmatrix}.$$

The characteristic polynomial for this matrix is $\lambda^4 - 30\lambda^2 + 102\lambda$.

Any polynomial can be factored into powers of linear and irreducible quadratic factors using real numbers. For example, consider the polynomial

$$\lambda^9 + 8\lambda^8 + 36\lambda^7 + 94\lambda^6 + 143\lambda^5 + 98\lambda^4 - 48\lambda^3 - 160\lambda^2 - 132\lambda - 40.$$

This polynomial factors into $(\lambda^2 + 2\lambda + 2)^2(\lambda^2 + 3\lambda + 10)(\lambda + 1)^2(\lambda - 1)$.

The **algebraic multiplicity** of an eigenvalue λ_0 is the number of times that the factor $(\lambda - \lambda_0)$ appears in the factorization of the characteristic polynomial. The **geometric multiplicity** of λ_0 is the dimension of its eigenspace.

Theorem. The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.

Example. Suppose $\mathbf{A}$ is a 9×9 matrix with characteristic polynomial

$$-\lambda^9 - 8\lambda^8 - 36\lambda^7 - 94\lambda^6 - 143\lambda^5 - 98\lambda^4 + 48\lambda^3 + 160\lambda^2 + 132\lambda + 40.$$

What can we say about the eigenspaces of $\mathbf{A}$?

Example. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

The diagonalization problem

A matrix $\mathbf{A}$ is diagonalizable if there exists a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

"Diagonalizing" a matrix has many applications. One is algebraic. If a matrix $\mathbf{A}$ is diagonalizable, then we can compute its powers $\mathbf{A}^k$ quickly. For example, if $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, then

$$\begin{aligned}\mathbf{A}^4 &= \left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)\left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)\left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)\left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)\\ &= \mathbf{P}\mathbf{D}^4\mathbf{P}^{-1}.\end{aligned}$$

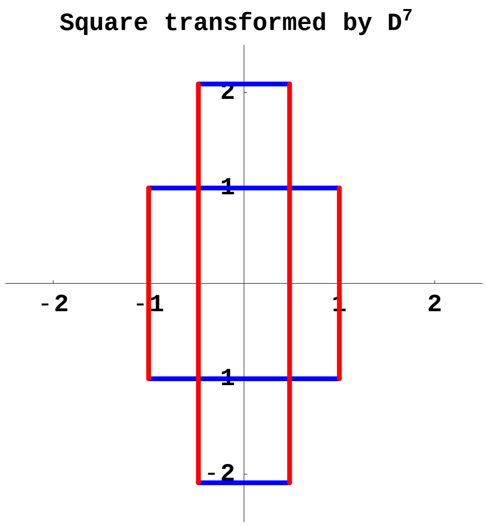
Another application is geometric. As we shall see, the matrix

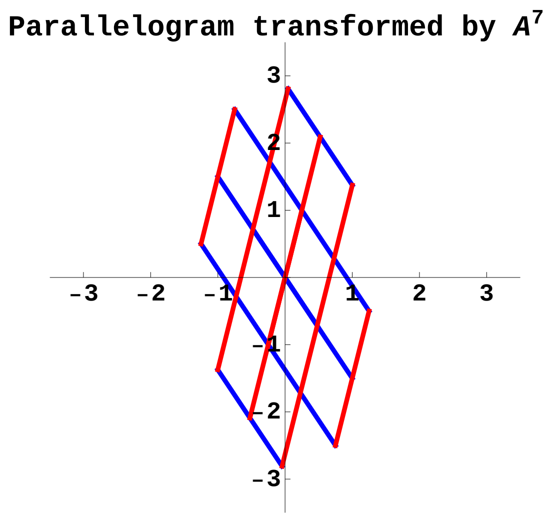
$$\mathbf{A} = \begin{bmatrix} \frac{158}{165} & \frac{19}{495} \\ \frac{38}{165} & \frac{1043}{990} \end{bmatrix}$$

is diagonalizable, and the corresponding diagonal matrix is

$$\mathbf{D} = \begin{bmatrix} \frac{9}{10} & 0 \\ 0 & \frac{10}{9} \end{bmatrix}.$$

There are animations on the course web page that illustrate how the matrix $\mathbf{A}$ transforms the plane in a way that is "similar" to the diagonal matrix $\mathbf{D}$.





Diagonalizing a matrix is a special case of the similarity problem.

Definition. Two square matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Note: We have already done two exercises related to similarity—Section 2.2 #8 and Section 3.2 #34.

Theorem. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are similar matrices. Then $\mathbf{A}$ and $\mathbf{B}$

1. have the same characteristic polynomial and consequently the same eigenvalues, and
2. the same geometric multiplicities for each eigenvalue.

Proof of 1:

Proof of 2:

What does this theorem say about matrices that can be diagonalized? In other words, if a matrix $\mathbf{A}$ can be diagonalized, what must $\mathbf{A}$ and $\mathbf{D}$ have in common?

Example. What can you say about a matrix $\mathbf{A}$ that is similar to the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}?$$

For an arbitrary matrix $\mathbf{A}$, what can be said about it if it is diagonalizable?

For example, can

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

be diagonalized?