More on the dot product

Definition. Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in $\mathbb{R}^n$, then $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

Theorem 1. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u} \geq 0$, and

(e) $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Definition. The length (or norm) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is $||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

Note that $||r\mathbf{v}|| = |r|\,||\mathbf{v}||$.

Given $\mathbf{v} \neq \mathbf{0}$, we normalize $\mathbf{v}$ by computing the vector $\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v}$.

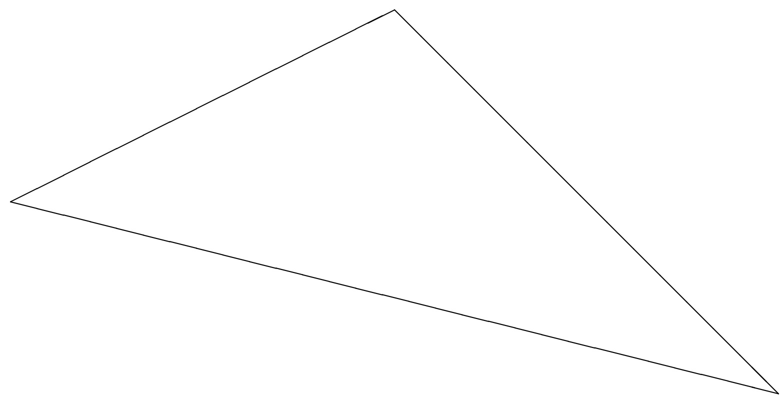
If we think of two vectors $\mathbf{u}$ and $\mathbf{v}$ as points in $\mathbb{R}^n$, then we define the distance between $\mathbf{u}$ and $\mathbf{v}$ as

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

What about angles? Let's start with right angles.

Definition. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$.

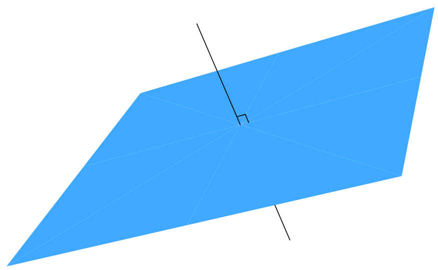
We can use the Law of Cosines to derive a more exact relationship between angles and the dot product.



Orthogonal complements

Given a subspace S of $\mathbb{R}^n$, we can consider the set of all vectors that are orthogonal to all vectors in S. For example, a plane through the origin in $\mathbb{R}^3$ can be described by one homogeneous linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$



Definition. Given a subspace S of $\mathbb{R}^n$, its orthogonal complement $S^\perp$ is the set

$$\{\mathbf{v} \mid \mathbf{v}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \text{ in } S\}.$$

Examples.

1. The orthogonal complement of a line through the origin in $\mathbb{R}^3$ is a plane through the origin.

2. The orthogonal complement of a plane through the origin in $\mathbb{R}^3$ is a line through the origin.

3. Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{bmatrix}.$$

   What can you say about a vector $\mathbf{v}$ in $\mathbb{R}^n$ that is orthogonal to all of the rows of $\mathbf{A}$?

Theorem. Let S be a subspace of $\mathbb{R}^n$ and let $S^\perp$ be its orthogonal complement. Then

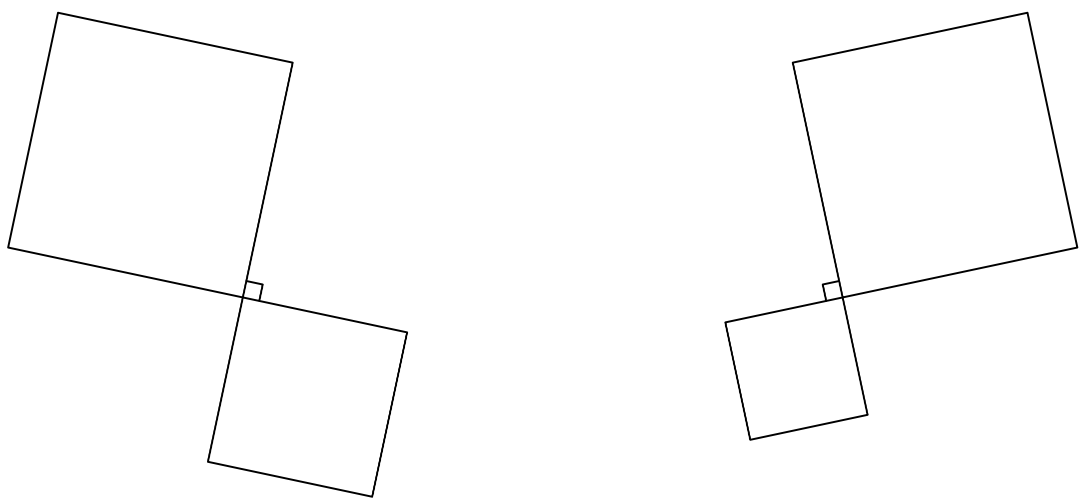
1. $S^\perp$ is a subspace of $\mathbb{R}^n$,
2. $\dim(S^\perp) = n - \dim(S)$,
3. $(S^\perp)^\perp = S$, and
4. every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is in S and $\mathbf{v}_2$ is in $S^\perp$.

It helps to have a little more theory before we verify properties 2, 3, and 4, but we can verify property 1 directly from the definition.

Solution sets of linear equations viewed in terms of Gilbert Strang's figure

Here's a diagram due to Gilbert Strang (MIT) that helps us understand the relationships among various subspaces associated to an $m \times n$ matrix $\mathbf{A}$. (Lay has a similar figure that illustrates the same idea at the top of page 335 of your text.)

Suppose that the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$ in $\mathbb{R}^m$.



This figure is one way of remembering the fact that the solution set of a consistent nonhomogeneous system consists of the sum of one particular solution of the nonhomogeneous system and the general solution of the associated homogeneous system (Theorem 6 in Chapter 1). It also suggests the second part of Exercise 23 in Section 6.3. That is, there is a unique $\mathbf{p}$ in Row $\mathbf{A}$ such that

$$\mathbf{Ap} = \mathbf{b}.$$

Orthogonal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$.

Example 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$

Theorem. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors.

1. If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$, then the weights c_i are given by $c_i = \dfrac{\mathbf{u} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$.

2. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.