Orthogonal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$.

Example 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$

Theorem. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors.

1. If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$, then the weights c_i are given by $c_i = \dfrac{\mathbf{u} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$.
2. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

Example. Using the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in Example 1, apply this theorem to the vector

$$\mathbf{u} = \begin{bmatrix} -3 \\ -2 \\ -5 \\ -9 \end{bmatrix}.$$

Orthonormal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is orthonormal if it is orthogonal and $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all i.

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

We can use matrices to express the fact that a set is orthogonal or orthonormal.

Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. The following three conditions are equivalent.

1. $\mathbf{A}^T = \mathbf{A}^{-1}$
2. The columns of $\mathbf{A}$ form an orthonormal basis of $\mathbb{R}^n$.
3. The rows of $\mathbf{A}$ form an orthonormal basis of $\mathbb{R}^n$.

Definition. Whenever a matrix satisfies the above theorem, it is said to be an orthogonal matrix.

Example. We can use the orthonormal basis of $\mathbb{R}^3$ given above to produce an orthogonal matrix.

Why are orthogonal matrices special?

Orthogonal projection

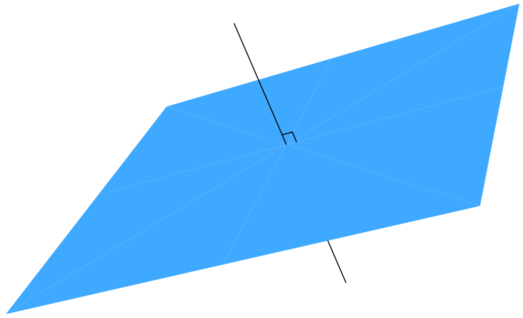
How do we project a vector $\mathbf{v}$ onto a subspace W?

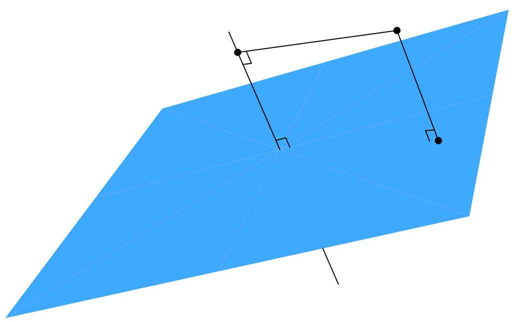
Theorem. (Orthogonal Decomposition Theorem)

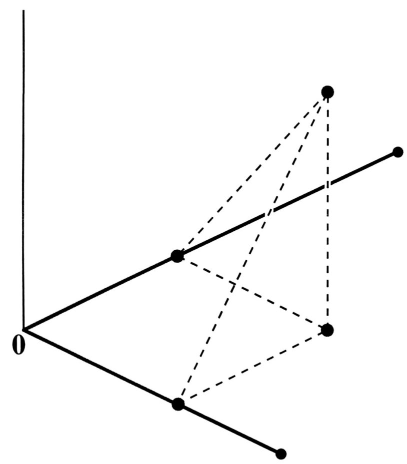
1. Each vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written uniquely as
$$\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp},$$
where $\mathbf{w}$ is in W and $\mathbf{w}^{\perp}$ is in $W^{\perp}$.

2. Given an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of W, then
$$\text{proj}_W \mathbf{v} \equiv \mathbf{w} = \left(\frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 + \ldots + \left(\frac{\mathbf{v} \cdot \mathbf{w}_k}{\mathbf{w}_k \cdot \mathbf{w}_k}\right) \mathbf{w}_k$$
and $\mathbf{w}^{\perp} = \mathbf{v} - \mathbf{w}$.

Note: Since the two vectors $\mathbf{w}$ and $\mathbf{w}^{\perp}$ are unique, they do not depend on the orthogonal basis of W that we use to compute them.







Example. Consider the orthogonal set

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$

Let W be Span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Compute $\text{proj}_W \mathbf{v}$ for

$$\mathbf{v} = \begin{bmatrix} -45 \\ -4 \\ 3 \\ 1 \end{bmatrix}.$$

Why is the Orthogonal Decomposition Theorem true?

Important consequence: If we want to find the distance of a vector $\mathbf{v}$ to a subspace W, then we compute

$$||\mathbf{w}^{\perp}|| = ||\mathbf{v} - \text{proj}_W \mathbf{v}||.$$

Example. Find the point closest to

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

in the subspace W spanned by the two vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$