Projection matrices

Theorem. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for a subspace W, then

$$\text{proj}_W \mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + \ldots + (\mathbf{v} \cdot \mathbf{u}_k)\mathbf{u}_k.$$

If

$$\mathbf{U} = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \quad \ldots \quad & \mathbf{u}_k \end{array} \right],$$

then $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T \mathbf{v}$.

The matrix $\mathbf{U}\mathbf{U}^T$ is called the *projection matrix* for the subspace W. It does not depend on the choice of orthonormal basis.

Example. Let's repeat the calculation I mentioned at the end of last class. Let

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \qquad \mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Using the Orthogonal Decomposition Theorem, we computed the projection of $\mathbf{v}$ onto $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$. We got

$$\text{proj}_W \mathbf{v} = 3\mathbf{w}_1 + \mathbf{w}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

Let

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{4}{\sqrt{26}} \\ -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{26}} \\ -\frac{1}{\sqrt{10}} & 0 \\ \frac{2}{\sqrt{10}} & \frac{3}{\sqrt{26}} \end{bmatrix}.$$

Then

$$\mathbf{U}\mathbf{U}^T = \mathbf{P} = \begin{bmatrix} \frac{93}{130} & -\frac{23}{65} & -\frac{1}{10} & -\frac{17}{65} \\ -\frac{23}{65} & \frac{57}{130} & \frac{1}{5} & -\frac{37}{130} \\ -\frac{1}{10} & \frac{1}{5} & \frac{1}{10} & -\frac{1}{5} \\ -\frac{17}{65} & -\frac{37}{130} & -\frac{1}{5} & \frac{97}{130} \end{bmatrix}.$$

Using the computer, we see that $\mathbf{P}^2 = \mathbf{P}$.

The Gram-Schmidt Process

This procedure produces an orthogonal (or orthonormal) basis from a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ of a subspace W. It is an inductive procedure.

We work with the subspaces

$$S_l = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_l\}.$$

The orthogonal basis for W based on this procedure applied to this basis is denoted $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$.

1. Let $\mathbf{v}_1 = \mathbf{x}_1$.
2. Let $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\,\mathbf{v}_1$.
3. Let $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\,\mathbf{v}_1 - \dfrac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\,\mathbf{v}_2$.

etc.

Example. Apply the Gram-Schmidt process to the basis

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Example. Let's compute the projection matrix $\mathbf{P}$ for orthogonal projection onto the plane $x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$.

What are the eigenvalues and eigenspaces of $\mathbf{P}$? (No computation required)