The matrix-vector product $\mathbf{Ax}$

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{x}$ be a vector in $\mathbb{R}^n$. We can define the product $\mathbf{Ax}$ as a linear combination of the vectors that come from the columns of $\mathbf{A}$.

Definition. Let $\mathbf{A}$ be an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \left[\begin{array}{c|c|c|c} & & & \\ \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_n \\ & & & \end{array} \right],$$

where $\mathbf{A}_k$ is the kth column of $\mathbf{A}$. Given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

in $\mathbb{R}^n$, we define the matrix-vector product $\mathbf{Ax}$ to be the linear combination

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \ldots + x_n\mathbf{A}_n.$$

Note that $\mathbf{Ax}$ is a vector in $\mathbb{R}^m$.

Example.

$$\begin{bmatrix} 3 & -8 \\ -1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-4)(3) + (2)(-8) \\ (-4)(-1) + (2)(5) \\ (-4)(2) + (2)(3) \end{bmatrix} = \begin{bmatrix} -28 \\ 14 \\ -2 \end{bmatrix}$$

Remark. Given an $m \times n$ matrix $\mathbf{A}$ and $\mathbf{x} \in \mathbb{R}^n$, then the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\left[\begin{array}{c|c|c|c|c} & & & & \\ \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_n & \mathbf{b} \\ & & & & \end{array} \right].$$

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution.
2. The columns of $\mathbf{A}$ span $\mathbb{R}^m$.
3. The matrix $\mathbf{A}$ has a pivot position in every row.

Warning: In this theorem, $\mathbf{A}$ is a *coefficient* matrix. The three statements are not equivalent if $\mathbf{A}$ is an augmented matrix.

Observation. Note that the kth entry in $\mathbf{Ax}$ is

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n.$$

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n$$

is called the **dot product** of $[\, a_{k1} \quad a_{k2} \quad \ldots \quad a_{kn} \,]$ and the vector $\mathbf{x}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. Then the matrix-vector product $\mathbf{Ax}$ is "linear" in $\mathbf{x}$. That is,

1. $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and
2. $\mathbf{A}(c\mathbf{u}) = c\mathbf{Au}$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and all c in $\mathbb{R}$.

Solution sets of systems of linear equations

Definition. Consider a linear system $\mathbf{Ax} = \mathbf{b}$. We say that it is *homogeneous* if $\mathbf{b} = \mathbf{0}$ and *nonhomogeneous* otherwise.

The homogeneous case $\mathbf{Ax} = \mathbf{0}$

Observation. Note that every homogeneous system is consistent. The solution $\mathbf{x} = \mathbf{0}$ is called the *trivial* solution. All other solutions are said to be nontrivial.

Theorem. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$, then any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is also a solution.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Express the solution set for $\mathbf{Ax} = \mathbf{0}$ as a span. (Note that $\mathbf{A}$ is a coefficient matrix, not an augmented matrix.)