

Solution sets of nonhomogeneous systems of linear equations

**Theorem.** Let  $\mathbf{p}$  be one solution to the nonhomogeneous equation  $\mathbf{Ax} = \mathbf{b}$  and let  $S$  be the set of all solutions to the associated homogeneous equation

$$\mathbf{Ax} = \mathbf{0}.$$

Then the solution set of  $\mathbf{Ax} = \mathbf{b}$  consists of all vectors in the set  $\mathbf{p} + S$ .

**Example.** Consider the linear system  $\mathbf{Ax} = \mathbf{b}$  whose augmented matrix is

$$\left[ \mathbf{A} \mid \mathbf{b} \right] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 9 \end{bmatrix}.$$

Linear independence

**Example.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

On Thursday, September 10, we determined that  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  is the plane  $P$  in  $\mathbb{R}^3$  given by the equation

$$x_1 + x_2 + x_3 = 0.$$

What happens to the span if we add a third vector  $\mathbf{v}_3$  to the set of vectors generating the span? In other words, how does  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  depend on the choice of  $\mathbf{v}_3$ ?

**Definition.** A (linear) dependence relation among a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is an equation of the form

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0},$$

where  $r_i \neq 0$  for some vector  $\mathbf{v}_i \in \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ .

**Example.**

$$2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Definition.** If there exists a dependence relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0}$$

among a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ , then we say that the set is *linearly dependent*. A set is *linearly independent* if it is not linearly dependent.

Matrix characterization

A dependence relation  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0}$  can be rewritten as the matrix equation  $\mathbf{A}\mathbf{r} = \mathbf{0}$  where

$$\mathbf{A} = \left[ \begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{array} \right] \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}.$$

Therefore, a dependence relation among the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is the same as a nontrivial solution to  $\mathbf{A}\mathbf{r} = \mathbf{0}$ .

**Example.** Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We can determine that these three vectors are linearly independent by considering the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

**Example.** Which of the following sets of vectors in  $\mathbb{R}^3$  are linearly independent? Why?

1.  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

2.  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$

3.  $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ \pi \end{bmatrix} \right\}$

4.  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

5.  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$

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**Theorem.** A nonzero set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  of vectors is linearly dependent if and only if, for some index  $j$ , the vector  $\mathbf{v}_j$  is a linear combination of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$ .

**Theorem.** If  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set of vectors in  $\mathbb{R}^n$ , then  $k \leq n$ .

**Example.** We know that the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$$

is linearly dependent. What are all possible dependence relations among this set of vectors?